REPORTS ON MATHEMATICAL LOGIC
34 (2000), 23–57

James G. RAFTERY, Clint J. VAN ALTEN

RESIDUATION IN COMMUTATIVE ORDERED MONOIDS WITH MINIMAL ZERO

1. Introduction

A *commutative pomonoid* is a structure $\mathbf{A} = \langle A; \oplus, 0; \leq \rangle$, whose reduct $\langle A; \oplus, 0 \rangle$ is a commutative monoid where $\leq$ is a partial order of A for which $\oplus$ is isotone in both of its arguments. We call $\mathbf{A}$ *residuated* provided that for any $x, y \in A$ there is a least $z \in A$ such that $x \leq z \oplus y$, in which case this z is denoted by $x \dot{-} y$ and the binary operation $\dot{-}$ on A is called *residuation.* In particular, such structures $\mathbf{A}$ satisfy $x \leq y \Leftrightarrow x \dot{-} y \leq 0$.

The abstract study of such pomonoids was inspired by the ideal lattices of commutative unital rings, with ideal multiplication, reversed set inclusion and the ring itself in the roles of $\oplus$, $\leq$ and 0. Here, although (unary) inverses are absent, residuation supplies a binary operation abstracting division. More recently, residuated commutative pomonoids in their full generality have received attention as natural models for fragments of lin-

1991 *Mathematics Subject Classification* 08C15, 06F05, 06F35, 03B22, 20M99.

Received 11 November 1999.

ear logic. In this generality, because of the presence of the relation $\leq$, they are not algebras.

This paper is a study of those commutative residuated pomonoids $\mathbf{A}$ (as above) that are *semi-integral*, by which we mean that the monoid identity 0 is a *minimal* (not necessarily the least) element of the poset $\langle A; \leq \rangle$. In this case $\mathbf{A}$ satisfies $x \leq y \Leftrightarrow x \dot{-} y \approx 0$. Consequently, $\mathbf{A}$ is first order definitionally equivalent to the *algebra* $\langle A; \oplus, \dot{-}, 0 \rangle$ of type $\langle 2, 2, 0 \rangle$, which we call a *sircomonoid.* These algebras form a quasivariety SIRCOM, and the methods of universal algebra become available for their investigation.

A well-understood class of sircomonoids is the quasivariety of *pocrims*: these are the *integral* sircomonoids, i.e., those in which the monoid identity is the (unique) *least* element of the order. Another is the variety of abelian groups $\langle G; +, -, 0 \rangle$ endowed with the discrete order.[1] As it happens, these two examples illuminate the residuation structure of sircomonoids $\mathbf{A} = \langle A; \oplus, \dot{-}, 0 \rangle$ in general. Indeed, $G = \{0 \dot{-} x : x \in A\}$ is the universe of an abelian group $\mathbf{G} = \langle G; +, \dot{-}, 0 \rangle$ and the function $\alpha : x \mapsto 0 \dot{-} (0 \dot{-} x)$ on A is idempotent and a homomorphism from $\mathbf{A}$ onto $\mathbf{G}$; also the congruence kernel of α is determined by $I = \alpha^{-1}[\{0\}]$, which is the universe of the largest subpocrim $\mathbf{I}$ of $\mathbf{A}$. Semi-integrality forces *all* elements of G to be minimal in $\langle A; \leq \rangle$; each element of A dominates just one of them. (See Figure 1 of Section 2.)

The above map α is not generally an *endomorphism* of $\mathbf{A}$ but it *is* always an endomorphism of the residuation *reduct* $\mathbf{A}^- = \langle A; \dot{-}, 0 \rangle$ of $\mathbf{A}$, i.e., α retracts $\mathbf{A}^-$ onto its *subalgebra* $\langle G; \dot{-}, 0 \rangle$. It is therefore profitable to consider the residuation subreducts (i.e., the subalgebras of the $\langle \dot{-}, 0 \rangle$-reducts) of sircomonoids in their own right. We prove that they are exactly Iséki's *BCI-algebras*, which form a quasivariety BCIA. The nontrivial part of this result, viz. that every BCI-algebra embeds in the residuation reduct of a sircomonoid, is an apparently new representation theorem for BCI-algebras; its proof yields as a special case Pałasinski, Ono-Komori and Fleischer's theorem that BCK-algebras are just the residuation subreducts of pocrims.

---

[1] Here, for convenience, $-$ is binary subtraction, rather than unary inversion.

The quasivarieties SIRCOM and BCIA are not varieties. We prove them to be relatively congruence extensile and that their relatively (or relatively finitely) subdirectly irreducible members are absolutely (or absolutely finitely) subdirectly irreducible, respectively.

SIRCOM and BCIA are relatively 0-regular and relatively congruence modular, but BCIA generates a variety that is neither congruence modular[2] nor congruence extensile. Nevertheless, we show that when K is SIRCOM or BCIA, the *relative* congruences of any $\mathbf{A} \in \mathsf{K}$ obey the *relative* commutator identity $[x, y]_{\mathsf{K}} \approx x \cap y \cap [1, 1]_{\mathsf{K}}$, (where 1 is A^2), that $[1, 1]_{\mathsf{K}} = [1, 1]$ and that $0/[1, 1]$ is the largest pocrim (or BCK-) subuniverse I of $\mathbf{A}$. The structure theorem $\mathbf{A}/I \cong \mathbf{G}$ is therefore an analogue of the fundamental theorem of abelian algebras (which fails in certain other relatively modular quasivarieties), and in K, the finitely subdirectly irreducible algebras are either affine or relatively prime. Moreover, we show that *finite* sircomonoids and BCI-algebras generate varieties *contained* in SIRCOM or BCIA, so results of Freese and McKenzie imply that any such variety is residually finite and finitely (equationally) based. In fact we show that the variety and quasivariety generated by a finite BCI-algebra coincide; this is not true of sircomonoids. We analyse subdirectly irreducible sircomonoids and develop a construction which allows us to prove that SIRCOM has continuously many subvarieties not generated only by pocrims and abelian groups; a similar result holds for BCIA.

Since every sircomonoid is an extension of its largest pocrim subalgebra $\mathbf{I}$ by an abelian group $\mathbf{G}$, it is of interest to characterize the 'disconnected' ones, i.e., those that decompose as $\mathbf{I} \times \mathbf{G}$. We prove that these are just the sircomonoids whose derived operation $x + y = x \dot{-} (0 \dot{-} y)$ is associative. The result yields a finite axiomatization of the quasivariety generated by all pocrims and all abelian groups, showing it to be a (proper) relative subvariety of SIRCOM. Analogous results hold for BCIA.

Finally, we consider the logical antecedents of BCI-algebras. We show that the 'assertional logic' of BCIA is a purely *axiomatic* extension of the

[2] The commutator theory of modular varieties therefore does not apply to it directly, while in the corresponding theory for relatively modular quasivarieties, the analogues of certain striking general results either fail or are open problems.

implication fragment **BCI** of Girard's linear logic, that it is the least 'pointedly algebraizable' simple extension of **BCI** and that, unlike **BCI**, it has a local deduction theorem.

A few notational preliminaries: For a binary relation ξ on a set S and $s \in S$, we denote by s/ξ the s-class $\{r \in S : \langle r, s\rangle \in \xi\}$ of ξ in S. For any poset $\mathbf{P} = \langle P; \leq\rangle$ and $Y \subseteq P$ and $p \in P$, we write $[Y)$ for the dually hereditary subset $\{q \in P : y \leq q \text{ for some } y \in Y\}$ of $\langle P; \leq\rangle$ generated by Y and we abbreviate $[\{p\})$ as $[p)$.

2. Sircomonoids and a Representation Theorem

A *sircomonoid* was defined in the introduction but let us give an intrinsic definition here: it is an algebra $\mathbf{B} = \langle B; \oplus, \dot{-}, 0\rangle$ of type $\langle 2, 2, 0\rangle$ such that for $a, b, c \in B$,

(i) $\langle B; \oplus, 0\rangle$ is a commutative monoid;
(ii) the relation $\leq$ defined by $a \leq b$ iff $a \dot{-} b = 0$ is a partial order of B;
(iii) (residuation) $a \dot{-} b \leq c$ iff $a \leq c \oplus b$;
(iv) (semi-integrality) $a \leq 0$ implies that $a = 0$.

In this case, we also have[3]

(v) (compatibility) $a \leq b$ and $c \leq d$ imply that $a \oplus c \leq b \oplus d$.

The class SIRCOM[4] of all sircomonoids therefore satisfies the following identities and quasi-identity (where our convention is that $x \dot{-} y \dot{-} z$ abbreviates $(x \dot{-} y) \dot{-} z$).

(M1) $x \dot{-} y \dot{-} z \approx x \dot{-} (z \oplus y)$,
(M2) $x \dot{-} 0 \approx x$,
(M3) $x \dot{-} x \approx 0$,

[3] Using (iii), from $(a \oplus c) \dot{-} c \leq a \leq b$, we infer $a \oplus c \leq b \oplus c$. Similarly, $c \leq d$ implies $c \oplus b \leq d \oplus b$, so (v) follows by commutativity of $\oplus$ and transitivity of $\leq$.

[4] acronym for semi-integral residuated commutative ordered monoid

(M4) $x \dot{-} y \dot{-} (x \dot{-} z) \dot{-} (z \dot{-} y) \approx 0$,

(M5) $(x \dot{-} y \approx 0 \ \textsf{and} \ y \dot{-} x \approx 0) \ \Rightarrow \ x \approx y$.

More strongly:

Proposition 1. *The class of all sircomonoids is just the quasivariety axiomatized by (M1) – (M5).*[5]

The proof is very similar to arguments [22],[39] for 'pocrims' (defined below) and a related class, and will be omitted. We shall show (Theorem 2 below) that the last four of the above properties axiomatize the class of residuation subreducts of sircomonoids. These four axioms have been the subject of much investigation in their own right, however: algebras of type $\langle 2, 0\rangle$ satisfying (M2) – (M5) are called *BCI-algebras*; they were introduced by K. Iséki in [21]. Let BCIA denote the quasivariety of all such algebras. Neither SIRCOM nor BCIA is a variety: see [17],[44].

Theorem 2. *Every BCI-algebra is a subalgebra of the reduct $\langle B; \dot{-}, 0\rangle$ of a sircomonoid $\mathbf{B} = \langle B; \oplus, \dot{-}, 0\rangle$. Thus, BCI-algebras are exactly the residuation subreducts of sircomonoids.*

A *pocrim* [5],[17] is a sircomonoid $\mathbf{B}$ in which 0 is the (unique) *least* element of the partial order $\leq$, i.e., $\mathbf{B}$ satisfies $0 \dot{-} x \approx 0$. A BCI-algebra satisfying this identity is called a *BCK-algebra*: these algebras also arose in [21] and they are exactly the residuation subreducts of pocrims (see [11],[33],[35]).

Before proving Theorem 2, we need some structural properties of BCI-algebras. It follows directly from the axioms (M2) – (M5) that the relation $\leq$ defined on a BCI-algebra $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ by the rule $x \leq y \Leftrightarrow x \dot{-} y = 0$ is a partial order of A with respect to which 0 is minimal in A, and that the binary operation $\dot{-}$ is isotone in its first argument and antitone in its second. The following identities are further well known consequences of

---

[5] (M3) is derivable from (M2) and (M4): $x \dot{-} x \approx x \dot{-} 0 \dot{-} (x \dot{-} 0) \dot{-} (0 \dot{-} 0) \approx 0$.

(M2) – (M5)[6]:

(1) $x \dot{-} y \dot{-} z \approx x \dot{-} z \dot{-} y$,
(2) $x \dot{-} (x \dot{-} y) \leq y$,
(3) $x \dot{-} (x \dot{-} (x \dot{-} y)) \approx x \dot{-} y$,
(4) $x \dot{-} y \dot{-} (z \dot{-} y) \leq x \dot{-} z$,
(5) $0 \dot{-} (x \dot{-} y) \approx 0 \dot{-} x \dot{-} (0 \dot{-} y)$.

Let $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ be a BCI-algebra. The function $\beta : A \to A$ defined by $\beta(x) = 0 \dot{-} x$ is an endomorphism of $\mathbf{A}$, by (5). Its image $G = \{0 \dot{-} x : x \in A\}$ is consequently the universe of a subalgebra $\mathbf{G}$ of $\mathbf{A}$. By (3), therefore, the map $\alpha = \beta^2$ is an *idempotent* endomorphism of $\mathbf{A}$, i.e., α retracts $\mathbf{A}$ onto $\mathbf{G}$. It follows that α and β have the same congruence kernel $\theta = \{\langle a, b\rangle \in A : 0 \dot{-} a = 0 \dot{-} b\}$, that G is a transversal of θ and that if $I = 0/\theta$ then $I \cap G = \{0\}$. (These observations are strengthened in Lemma 3 below.) Note that $I = \{a \in A : 0 \leq a\}$ and that I is the universe of a subalgebra $\mathbf{I}$ of $\mathbf{A}$, again by (5). Evidently $\mathbf{I}$ is the largest subalgebra of $\mathbf{A}$ that is a BCK-algebra. (Moreover, if $\mathbf{A}$ is the residuation *reduct* of a sircomonoid $\mathbf{A}^+$ then I is closed under $\oplus$, by (M1), so $\mathbf{I}$ is then the residuation reduct of the largest pocrim subalgebra of $\mathbf{A}^+$.)

The subalgebra $\mathbf{G}$ of $\mathbf{A}$ is the subtraction reduct of an abelian group $\mathbf{G}^+ = \langle G; +, \dot{-}, 0\rangle$ where for $a, b \in G$, we define $a + b = a \dot{-} (0 \dot{-} b)$. (A proof of this can be found in [30, Theorem 3]) The unary inversion is given by $-a = 0 \dot{-} a$, so $\dot{-}$ is the binary subtraction operation. (For convenience, we take groups to have type $\langle 2, 2, 0\rangle$, rather than $\langle 2, 1, 0\rangle$.) The restriction of $\leq$ to G is the discrete order. We adopt the practice of writing $-c$ for $0 \dot{-} c$ when $c \in G$ but *not* when $c \in A \setminus G$. In the former case, of course, $-(-c) = c$.

When $\mathbf{A}$ is the residuation reduct of a sircomonoid $\mathbf{A}^+ = \langle A; \oplus, \dot{-}, 0\rangle$, the operations $+$ and $\oplus$ need not coincide on G; indeed, G need not

[6] To derive (5), note first that $0 \dot{-} y \approx x \dot{-} y \dot{-} x$ (by (M3), (1)), whence $0 \dot{-} x \dot{-} (0 \dot{-} y) \approx 0 \dot{-} x \dot{-} (x \dot{-} y \dot{-} x) \leq 0 \dot{-} (x \dot{-} y)$ (by (4)). Then since $0 \approx 0 \dot{-} x \dot{-} (0 \dot{-} x)$, we have $0 \dot{-} (0 \dot{-} x \dot{-} (0 \dot{-} y)) \approx 0 \dot{-} x \dot{-} (0 \dot{-} x) \dot{-} (0 \dot{-} x \dot{-} (0 \dot{-} y)) \leq 0 \dot{-} y \dot{-} (0 \dot{-} x) \leq x \dot{-} y$ (by (M4)), whence $0 \dot{-} (x \dot{-} y) \dot{-} (0 \dot{-} x \dot{-} (0 \dot{-} y)) \approx 0 \dot{-} (0 \dot{-} x \dot{-} (0 \dot{-} y)) \dot{-} (x \dot{-} y) \approx 0$, i.e., $0 \dot{-} (x \dot{-} y) \leq 0 \dot{-} x \dot{-} (0 \dot{-} y)$.

be closed under $\oplus$. In this case α is an epimorphism[7] from $\mathbf{A}^+$ to the group $\mathbf{G}^+$ (so θ is a congruence of $\mathbf{A}^+$ and $\mathbf{A}^+/\theta \cong \mathbf{G}^+$) but α is not an endomorphism of $\mathbf{A}^+$.

Parts (i), (ii) and (iii) of the next lemma are proved in [32].

Lemma 3.

(i) *θ is the transitive closure of the relation $\leq \cup (\leq^{-1})$ on A.*

(ii) *If $a \in A$ then $a/\theta = -(0 \dot{-} a)/\theta = [-(0 \dot{-} a))$, so $-(0 \dot{-} a) \in G$ is the unique minimal element g of $\langle A; \leq \rangle$ such that $g \leq a$. In particular, G is the set of all minimal elements of $\langle A; \leq \rangle$.*

(iii) *If $a \in A$ and $g \in G$ then $g \dot{-} a = 0 \dot{-} (a \dot{-} g) \in G$.*

(iv) *If $g_1, g_2 \in G$ and $g_1 \leq a_1 \in A$ and $g_2 \leq a_2 \in A$ then $g_1 \dot{-} g_2 \leq a_1 \dot{-} a_2$. If in addition, $\mathbf{A}$ is the residuation reduct of a sircomonoid $\langle A; \oplus, \dot{-}, 0 \rangle$ then $g_1 + g_2 \leq a_1 \oplus a_2$.* (See Figure 1 below.)

(v) *If $I = \{0\}$ then $\mathbf{A} = \mathbf{G}$.*

Proof. To prove the first assertion of (iv), note that $g_1 \dot{-} g_2 \dot{-} (a_1 \dot{-} a_2) \in 0/\theta \cap G$ (by (i) and (iii)) $= \{0\}$. For the second, observe that

$$(g_1 + g_2) \dot{-} (a_1 \oplus a_2) = g_1 \dot{-} (-g_2) \dot{-} a_2 \dot{-} a_1$$
$$= g_1 \dot{-} a_1 \dot{-} (-g_2) \dot{-} a_2 = -(-g_2) \dot{-} a_2 = g_2 \dot{-} a_2 = 0.$$

Note that (v) follows from (ii), (M3) and (M5).

For a BCI-algebra $\mathbf{A}$ as above, an element $b \in A$ and a finite nonempty sequence $\vec{a} = a_1, \ldots, a_n$ of elements of A, define $b \dot{-} \vec{a} = b \dot{-} a_1 \dot{-} \ldots \dot{-} a_n$ and

$$J(\vec{a}) = \{x \in A : x \dot{-} \vec{a} = 0\}$$

and let $J(\mathbf{A})$ be the set of all such $J(\vec{a})$. Observe that each $J(\vec{a})$ is nonempty, e.g., $x \dot{-} (x \dot{-} \vec{a}) \in J(\vec{a})$ for all $x \in A$, by (1) and (M3). Define a binary operation $*$ on $J(\mathbf{A})$ by

$$J(a_1, \ldots, a_n) * J(b_1, \ldots, b_m) = J(a_1, \ldots, a_n, b_1, \ldots, b_m).$$

[7] Using (3), (5) and (M1), we have $(-(0 \dot{-} x)) + (-(0 \dot{-} y)) \approx (-(0 \dot{-} x)) \dot{-} (0 \dot{-} y) \approx 0 \dot{-} (0 \dot{-} x \dot{-} y) \approx -(0 \dot{-} (y \oplus x))$.

Let $\mathbf{J}(\mathbf{A}) = \langle J(\mathbf{A}); *, \{0\}; \subseteq\rangle$.

Parts (i), (iii) and (iv) of the following lemma were proved by Huang [20] but to see that Huang's construction is the same as the one presented here, one needs to note that for $J(\vec{a}), J(\vec{b}) \in J(\mathbf{A})$,

$$J(\vec{a}) \subseteq J(\vec{b}) \text{ iff } x \dot{-} \vec{b} \dot{-} (x \dot{-} \vec{a}) = 0, \text{ for all } x \in A.$$

(Sufficiency is obvious. For necessity, use the fact that $x \dot{-} (x \dot{-} \vec{a}) \in J(\vec{a})$.) From this it follows that $J(a) \subseteq J(\vec{b})$ whenever $a \in J(\vec{b})$.

Lemma 4.

(i) $\mathbf{J}(\mathbf{A})$ *is a commutative pomonoid.*

(ii) $\{\{g\} : g \in G\}$ *is the set of minimal elements of* $\langle J(\mathbf{A}); \subseteq\rangle$*; in particular,* $\{0\}$ *is a minimal element and every* $J(\vec{c}) \in J(\mathbf{A})$ *contains a unique minimal element, viz.* $J(-(0 \dot{-} \vec{c}))$.

(iii) *For any* $a \in A$, *and* $J(\vec{b}) \in J(\mathbf{A})$, *we have* $J(a \dot{-} \vec{b}) = \bigcap\{J(\vec{c}) \in J(\mathbf{A}) : J(a) \subseteq J(\vec{c}) * J(\vec{b})\}$.

(iv) *The map* $\eta_{\mathbf{A}} : a \mapsto J(a)$ $(a \in A)$ *is an isotone embedding of* $\langle A; \leq\rangle$ *into* $\langle J(\mathbf{A}); \subseteq\rangle$ *with* $\eta_{\mathbf{A}}(0) = J(0) = \{0\}$.

(v) *Let* $g_1, g_2 \in G$, *and let* $J(\vec{a}), J(\vec{b}) \in J(\mathbf{A})$ *with* $g_1 \in J(\vec{a})$ *and* $g_2 \in J(\vec{b})$. *Then* $g_1 + g_2$ $(= g_1 \dot{-} (0 \dot{-} g_2)) \in J(\vec{a}) * J(\vec{b})$. *If* $g_1 \leq a \in A$ *then* $g_1 \dot{-} g_2 \in J(a \dot{-} \vec{b})$.

Proof. (i) and (iv) are straightforward. For (iii), see [20] Theorem 2.1. In (ii), observe first that, for each $g \in G$, $\{g\} = J(g)$ (Lemma 3(ii)) is a minimal element of $\langle J(\mathbf{A}); \subseteq\rangle$. For the converse, note that if $x \in J(\vec{c})$ then $-(0 \dot{-} x) \in J(\vec{c})$ (by (2)), so $J(-(0 \dot{-} x)) \subseteq J(\vec{c})$, and $-(0 \dot{-} x) \in G$. If $J(\vec{c})$ is also minimal then $J(-(0 \dot{-} x)) = J(\vec{c})$. For uniqueness, suppose that $J(0 \dot{-} y), J(0 \dot{-} z) \subseteq J(\vec{c})$, so $0 \dot{-} y \dot{-} \vec{c} = 0 = 0 \dot{-} z \dot{-} \vec{c}$, i.e., $0 \dot{-} \vec{c} \leq y, z$. Then $-(0 \dot{-} \vec{c}) \geq 0 \dot{-} y, 0 \dot{-} z$, and all three of these are in G, so they are equal. Under the assumptions of (v),

$$(g_1 + g_2) \dot{-} \vec{a} \dot{-} \vec{b} = g_1 \dot{-} \vec{a} \dot{-} (-g_2) \dot{-} \vec{b} = (-(-g_2)) \dot{-} \vec{b} = g_2 \dot{-} \vec{b} = 0$$

and if $J(a) \subseteq J(\vec{c}, \vec{b})$ then $g_1 \dot{-} \vec{c} \dot{-} \vec{b} = 0$ (because $g_1 \dot{-} a = 0$); now

$$g_1 \dot{-} g_2 \dot{-} \vec{c} \;=\; g_1 \dot{-} \vec{c} \dot{-} g_2 \;\geq\; g_1 \dot{-} \vec{c} \dot{-} \vec{b} \dot{-} (g_2 \dot{-} \vec{b}) \;=\; 0 \dot{-} 0 \;=\; 0$$

but $g_1 \dot{-} g_2 \dot{-} \vec{c} \in G$ (by Lemma 3(iii)) so $g_1 \dot{-} g_2 \dot{-} \vec{c} = 0$, i.e., $g_1 \dot{-} g_2 \in J(\vec{c})$. By (iii), $g_1 \dot{-} g_2 \in J(a \dot{-} \vec{b})$.

Although (iii) says that the images of elements of A under the embedding $\eta_{\mathbf{A}}$ are 'residuable' in $\mathbf{J}(\mathbf{A})$, the pomonoid $\mathbf{J}(\mathbf{A})$ is not fully residuated. To fill this gap we require a further embedding construction. At this point a naive application to a BCI-algebra $\mathbf{A}$ of the constructions in [33] and [11] (which would embed $\mathbf{J}(\mathbf{A})$ into the dual of the lattice of its order filters, endowed with a suitable monoid operation) yields a containing residuated commutative pomonoid $\mathbf{M}$ whose zero is *not* minimal. The following construction and lemma show how to choose $\mathbf{M}$ differently so that it has a minimal zero.

Let $\mathbf{B} = \langle B; *, 0^{\mathbf{B}}; \leq \rangle$ be a commutative pomonoid. Take an abelian group $\mathbf{G} = \langle G; +, -, 0^{\mathbf{G}} \rangle$ with $0 = 0^{\mathbf{B}} = 0^{\mathbf{G}} \in G \subseteq B$. Assume that for each $b \in B$ there is a unique $g \in G$ with $g \leq b$, so G is the set of minimal elements of $\langle B; \leq \rangle$. Also assume that for any $g_1, g_2 \in G$ and $b_1, b_2 \in B$,

$$\text{if } g_1 \leq b_1 \text{ and } g_2 \leq b_2 \text{ then } g_1 + g_2 \leq b_1 * b_2. \tag{6}$$

For each $g \in G$, let C_g be the set of all subsets of $[g) = \{b \in B : g \leq b\}$ that are dually hereditary (i.e., upward closed) in $\langle B; \leq \rangle$ and let $C = \bigcup_{g \in G} C_g$. For $S, T \in C$, define $S \oplus T = [\{a * b : a \in S \text{ and } b \in T\})$. By (6), $\oplus$ is a binary operation on C.

Lemma 5. *The structure $\langle C; \oplus, [0); \supseteq \rangle$ is a commutative residuated pomonoid with minimal zero. In its equivalent sircomonoid $\mathbf{C} = \mathbf{C}(\mathbf{B}) = \langle C; \oplus, \dot{-}, [0) \rangle$, we have $S \dot{-} T = [\{b \in B : b * T \subseteq S\})$ for $S, T \in C$, where $b * T = \{b * t : t \in T\}$. The map $\chi_{\mathbf{B}} : b \mapsto [b)$ $(b \in B)$ from $\mathbf{B}$ to $\langle C; \oplus, [0); \supseteq \rangle$ is an isotone monoid embedding that preserves existing residuals, i.e., whenever $a, b \in B$ and c is the least element of B such that $a \leq c * b$ then $\chi_{\mathbf{B}}(c) = \chi_{\mathbf{B}}(a) \dot{-} \chi_{\mathbf{B}}(b)$.*

The proof of the lemma is straightforward. The crucial facts are that every element of C is contained in $[g)$ for a unique $g \in G$ and that, since $\mathbf{G}$ is a group, whenever $S \in C_g$ and $T \in C_h$ and $U \in C_k$ with $U \oplus T \subseteq S$ then $k = g - h$.

For any BCI-algebra $\mathbf{A}$, by combining Lemmas 4 and 5, we obtain a $\langle \dot{-}, 0\rangle$-embedding $\chi_{\mathbf{J}(\mathbf{A})} \circ \eta_{\mathbf{A}} : a \mapsto \{J(\vec{c}) \in J(\mathbf{A}) : J(a) \subseteq J(\vec{c})\}$ $(a \in A)$ of $\mathbf{A}$ into the residuation reduct of the sircomonoid $\mathbf{C}(\mathbf{J}(\mathbf{A}))$. This completes the proof of Theorem 2. In the case that 0 is the *least* element of a BCI-algebra $\mathbf{A}$ (i.e., $G = \{0\}$ and $A = I$), the above construction yields the result of Pałasinski [35], Ono-Komori [33] and Fleischer [11] that BCK-algebras are just the residuation subreducts of pocrims.

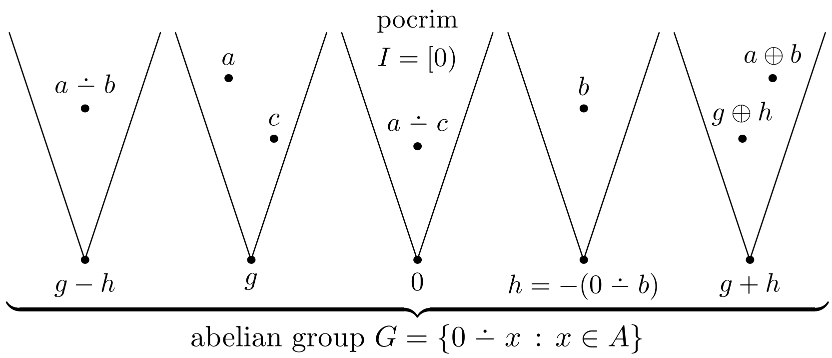

Figure 1. The structure of a sircomonoid $\mathbf{A}$

3. Algebraic Analysis of Sircomonoids

Let $\mathcal{L}$ be a language of algebras and K a quasivariety of $\mathcal{L}$-algebras. A congruence θ of an $\mathcal{L}$-algebra $\mathbf{A}$ (with universe A) is called a K-*congruence* (or *relative congruence*) of $\mathbf{A}$ if $\mathbf{A}/\theta \in \mathsf{K}$. We denote by $\operatorname{Con}_{\mathsf{K}} \mathbf{A}$ the set of all K-congruences of $\mathbf{A}$. This set becomes an algebraic lattice $\mathbf{Con}_{\mathsf{K}}\, \mathbf{A}$ when ordered by inclusion. (As such it coincides with the full congruence lattice $\mathbf{Con}\, \mathbf{A}$ if K is a variety and $\mathbf{A} \in \mathsf{K}$.) The least congruence and K-congruence of $\mathbf{A}$ containing $X \subseteq A^2$ are denoted, respectively, by $\Theta^{\mathbf{A}}(X)$ and $\Theta^{\mathbf{A}}_{\mathsf{K}}(X)$.

Let $\mathbf{A} \in \mathsf{K}$. We call $\mathbf{A}$, K-*subdirectly irreducible* (or *relatively subdirectly irreducible*) if $\mathbf{A}$ has a smallest nonidentity K-congruence. Every algebra in K is a subdirect product of K-subdirectly irreducible algebras *in* K. We call $\mathbf{A}$ *finitely* K-*subdirectly irreducible* if the identity congruence of $\mathbf{A}$ is meet irreducible in $\mathbf{Con}_{\mathsf{K}}\,\mathbf{A}$. A [finitely] K-subdirectly irreducible algebra need not be [finitely] subdirectly irreducible in the absolute sense. We say that $\mathbf{A}$ is K-*simple* if it has just two K-congruences.

We call K *relatively congruence modular* [resp. *distributive*] if the lattice $\mathbf{Con}_{\mathsf{K}}\,\mathbf{A}$ is modular [resp. distributive] for all $\mathbf{A} \in \mathsf{K}$. We say that K has the *relative congruence extension property* (RCEP) if for any K-congruence ξ of a subalgebra $\mathbf{B}$ of any $\mathbf{A} \in \mathsf{K}$ we have $B^2 \cap \Theta_{\mathsf{K}}^{\mathbf{A}}(\xi) = \xi$.

We call K *relatively* 0*-regular* if 0 is a constant symbol of $\mathcal{L}$ and the K-congruences of algebras $\mathbf{A}$ in K are determined by their 0-classes, i.e., whenever $\xi, \zeta \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ with $0^{\mathbf{A}}/\xi = 0^{\mathbf{A}}/\zeta$ then $\xi = \zeta$. Such quasivarieties K are characterized[8] by the existence of binary $\mathcal{L}$-terms d_j, $j < m \in \omega$, such that

$$\mathsf{K} \models (\textstyle\bigwedge_{j<m} d_j(x, y) \approx 0) \;\Leftrightarrow\; x \approx y. \tag{7}$$

In this case the 0-classes of relative congruences of any algebra $\mathbf{A} \in \mathsf{K}$ also form an algebraic lattice onto which the map $\xi \mapsto 0^{\mathbf{A}}/\xi$ ($\xi \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$) is a lattice isomorphism.[9] The inverse isomorphism is given by $J \mapsto \{\langle a, b\rangle \in A^2 : d_j(a, b) \in J \text{ for all } j < m\}$. A relatively 0-regular quasivariety need not be relatively congruence modular (e.g., [7]) but 0-regular varieties are congruence modular [14].

It is well known that the quasivarieties of pocrims and of BCK-algebras are relatively congruence distributive. Note that SIRCOM and BCIA are not relatively distributive, as they contain varieties of abelian groups (up to term equivalence).

3.1. Closed Ideals. Now let $\mathbf{A}$ be a sircomonoid (with $\mathsf{K} = \mathsf{SIRCOM}$) or a BCI-algebra (with $\mathsf{K} = \mathsf{BCIA}$); let $0 = 0^{\mathbf{A}}$. A subset J of A is called a *closed ideal*[10] of $\mathbf{A}$ if $J = 0/\xi$ for some $\xi \in \mathbf{Con}_{\mathsf{K}}\,\mathbf{A}$. It is known (e.g., see [29]) that the closed ideals of $\mathbf{A}$ are just the nonempty subsets J of A such

[8] For varieties, this characterization of 0-regularity appears in [9],[10].

[9] See [6] for a proof and internal characterization of the 0-classes.

[10] The literature on BCI-algebras reserves the term *ideal* for a more general kind of subset.

that for any $a, b \in A$, if $b, a \dot{-} b \in J$ then (i) $a \in J$ and (ii) $0 \dot{-} b \in J$. It follows that a closed ideal of $\mathbf{A}$ is both a subuniverse of $\mathbf{A}$ and a hereditary subset of $\langle A; \leq \rangle$. By (M3) and (M5), K is relatively 0-regular, so the set $\mathrm{CId}\,\mathbf{A}$ of all closed ideals of $\mathbf{A}$ is the universe of an algebraic lattice $\mathbf{CId}\,\mathbf{A}$ (ordered by inclusion) and the map $\xi \mapsto 0/\xi$ ($\xi \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$) is a lattice isomorphism from $\mathbf{Con}_{\mathsf{K}}\,\mathbf{A}$ onto $\mathbf{CId}\,\mathbf{A}$, with inverse isomorphism given by $J \mapsto \xi(J) := \{\langle a, b \rangle \in A^2 : a \dot{-} b, b \dot{-} a \in J\}$ ($J \in \mathrm{CId}\,\mathbf{A}$). We abbreviate the factor algebra $\mathbf{A}/\xi(J) \in \mathsf{K}$ as $\mathbf{A}/J$. For $Y \subseteq A$, denote the closed ideal of $\mathbf{A}$ generated by Y (i.e., the intersection of all closed ideals of $\mathbf{A}$ that contain Y) as $\mathrm{Cig}^{\mathbf{A}}(Y)$.

Lemma 6. *Let $\mathbf{A}$ be a sircomonoid or BCI-algebra and $Y \subseteq A$. Let $Z = \{a \in A : \exists\, n \in \omega$ and $\exists\, b_1, \ldots, b_n \in Y \cup (0 \dot{-} Y)$ such that $a \dot{-} \vec{b} = 0\}$, where $0 \dot{-} Y = \{0 \dot{-} y : y \in Y\}$. Then $Z = \mathrm{Cig}^{\mathbf{A}}(Y)$.*[11]

Proof. The only nontrivial task is checking that Z is a closed ideal of $\mathbf{A}$. Certainly $0 \in Z$ (e.g., choose $n = 0$). Let $a \in A$ and $b, a \dot{-} b \in Z$. Then there exist finite (possibly empty) sequences $\vec{c}, \vec{d}$ of elements of $Y \cup (0 \dot{-} Y)$ with $b \dot{-} \vec{c} = 0 = a \dot{-} b \dot{-} \vec{d} = a \dot{-} \vec{d} \dot{-} b$. Now

$$a \dot{-} \vec{d} \dot{-} \vec{c} = a \dot{-} \vec{d} \dot{-} \vec{c} \dot{-} (b \dot{-} \vec{c}) \dot{-} (a \dot{-} \vec{d} \dot{-} b) = 0, \quad \text{by (4)},$$

so $a \in Z$. Suppose $\vec{c}$ is $c_1, \ldots, c_m$. Then, by (5),

$$0 \dot{-} b \dot{-} (0 \dot{-} c_1) \dot{-} \ldots \dot{-} (0 \dot{-} c_m) = 0 \dot{-} (b \dot{-} \vec{c}) = 0 \dot{-} 0 = 0. \qquad (8)$$

If $c_i \in Y$ then $0 \dot{-} c_i \in 0 \dot{-} Y$. If $c_i = 0 \dot{-} y$ for some $y \in Y$ then, by (2), the equation (8) remains true when we replace $0 \dot{-} c_i$ by y. Thus, $0 \dot{-} b \in Z$, as required.

Corollary 7. *Both the quasivariety of sircomonoids and the quasivariety of BCI-algebras have the relative congruence extension property. Thus, any of their subvarieties has the congruence extension property (CEP).*

Proof. Let $\mathbf{A}$ be a sircomonoid or a BCI-algebra. By relative 0-regularity it suffices to show that for any subalgebra $\mathbf{B}$ of $\mathbf{A}$ and any

[11] When $n = 0$ (e.g. when $Y = \emptyset$) interpret $a \dot{-} \vec{b}$ as a. Thus, $\mathrm{Cig}^{\mathbf{A}}(\emptyset) = \{0\}$.

closed ideal J of $\mathbf{B}$, we have $B \cap \mathrm{Cig}^{\mathbf{A}}(J) = J$. This follows directly from the previous lemma.

Let $\mathbf{A}$ be a BCI-algebra or sircomonoid and $a, b \in A$. Define $b \dot{-} (0a) = b$ and, for $n \in \omega$, $b \dot{-} ((n+1)a) = b \dot{-} (na) \dot{-} a$. If $\mathbf{A}$ is a sircomonoid, define $0a = 0^{\mathbf{A}}$ and, for $n \in \omega$, $(n+1)a = (na) \oplus a$. (Thus, expressions like $a \dot{-} (nb)$ are unambiguous in sircomonoids.) As a special case of Lemma 6, we have that $b \in \mathrm{Cig}^{\mathbf{A}}(\{a\})$ if and only if there exist $n, m \in \omega$ such that $b \dot{-} (na) \dot{-} (m(0 \dot{-} a)) = 0$ (which is equivalent, in a sircomonoid, to $b \leq (na) \oplus (m(0 \dot{-} a)))$. In particular, for $g \in G = \{0 \dot{-} a : a \in A\}$, if $J = \mathrm{Cig}^{\mathbf{A}}(\{g\})$ then $J \cap G$ is the subgroup of $\langle G; +, \dot{-}, 0\rangle$ generated by g and if $I = [0)$ then, by Lemma 3, $J \cap I = \{i \in I : i \dot{-} (ng) \dot{-} (n(-g)) = 0$ for some $n \in \omega\}$. (In a sircomonoid, $J \cap I = \{i \in I : i \leq n(g \oplus (-g))$ for some $n \in \omega\}$.)

Lemma 8. *Let* $\mathbf{A}$ *be a sircomonoid or a BCI-algebra and let* $J \subseteq A$. *Then* J *is a closed ideal of* $\mathbf{A}$ *if and only if the following is true: for any* $\langle \dot{-}, 0\rangle$ *term*

$$t(x_1, \ldots, x_m, y_1, \ldots, y_n) = t(\vec{x}, \vec{y}) \ \ (m, n \in \omega),$$

if all BCI-algebras satisfy $t(\vec{x}, \vec{0}) \approx 0$, *and* $\vec{a}$ *is* $a_1, \ldots, a_m \in A$ *and* $\vec{c}$ *is* $c_1, \ldots, c_n \in J$ *then* $t(\vec{a}, \vec{c}) \in J$.[12]

Proof. Necessity: Note that if τ is any reflexive binary relation on A that is a subuniverse of $\mathbf{A}^2$ (e.g., any congruence or relative congruence of $\mathbf{A}$) then $0/\tau$ satisfies the lemma's condition on terms. In particular, this is true of the closed ideals of $\mathbf{A}$. Sufficiency: Let $J \subseteq A$ satisfy the condition on terms. Using the nullary term 0, we infer that $0 \in J$. Let $a \in A$ and $b, a \dot{-} b \in J$. Then $0 \dot{-} b \in J$ because BCIA satisfies $t(0) \approx 0$, where $t(y)$ is $0 \dot{-} y$. Finally, if $t(x, y_1, y_2)$ is $x \dot{-} (x \dot{-} y_1 \dot{-} y_2)$ then BCIA satisfies $t(x, 0, 0) \approx 0$, so $a = a \dot{-} 0 = a \dot{-} (a \dot{-} (a \dot{-} b) \dot{-} b) = t(a, a \dot{-} b, b) \in J$. Thus, J is a closed ideal of $\mathbf{A}$.

[12] This shows that for sircomonoids and BCI-algebras, the notion of closed ideal coincides with that of 'ideal' studied by A. Ursini [13] for algebras with a 0.

The next two corollaries are derived from this lemma exactly as for pocrims and related quasivarieties: see, e.g., [39, Lemma 12] and [41, Corollary 3.4], respectively.

Corollary 9. *Let* K *be* SIRCOM *or* BCIA *and* $\mathbf{A} \in \mathsf{K}$*. For any reflexive binary relation* τ *on* A *that is a subuniverse of* $\mathbf{A}^2$ *(e.g., any congruence of* $\mathbf{A}$*) we have* $\Theta^{\mathbf{A}}_{\mathsf{K}}(\tau) = \{\langle a, b\rangle \in A^2 : a \dot{-} b, b \dot{-} a \in 0/\tau\}$*, hence* $0/\tau = 0/\Theta^{\mathbf{A}}_{\mathsf{K}}(\tau)$.

For a class K of $\mathcal{L}$-algebras, let $\mathrm{V}(\mathsf{K})$ and $\mathsf{Q}(\mathsf{K})$ denote, respectively, the variety and the quasivariety generated by K. Thus for a quasivariety K, $\mathrm{V}(\mathsf{K})$ is the class of all homomorphic images of members of K.

Corollary 10. *Let* K *be* SIRCOM *or* BCIA*. A quasi-identity of the form*

$$\left(\bigwedge\nolimits_{i<n} s_i(\vec{x}) \approx 0\right) \;\Rightarrow\; t(\vec{x}) \approx 0$$

is satisfied by K *iff it is satisfied by the variety* $\mathrm{V}(\mathsf{K})$.

Corollary 11. *Let* K *be* SIRCOM *or* BCIA *and* $\mathbf{A} \in \mathsf{K}$*. Then* K *is relatively congruence modular. If* $\mathbf{A}$ *is* K*-subdirectly irreducible* [*resp. finitely* K*-subdirectly irreducible;* K*-simple*] *then it is subdirectly irreducible* [*resp. finitely subdirectly irreducible; simple*]*. Thus, every algebra in* K *is a subdirect product of (absolutely) subdirectly irreducible algebras in* K*.*

Proof. Any relatively 0-regular quasivariety K for which the conclusion of Corollary 10 is true has the properties asserted in the statement of the present corollary: see [8].

Therefore, sircomonoids and BCI-algebras have modular lattices of closed ideals. For BCI-algebras, this was proved directly in [42, Theorem 7]. Henceforth we abbreviate '[finitely] subdirectly irreducible' as [F]SI.

3.2. Subdirect Irreducibility. Let K be SIRCOM or BCIA. If $\mathbf{A} \in \mathsf{K}$ is SI with least nonidentity congruence ξ then, although ξ and $\Theta^{\mathbf{A}}_{\mathsf{K}}(\xi)$ need not coincide, they have the same 0-class, and this is the smallest nonzero closed ideal of $\mathbf{A}$. Conversely, if $\mathbf{A}$ has a least nonzero closed ideal J then $\mathbf{A}$ is SI; in this case we call J the *monolith* of $\mathbf{A}$.

Let $\mathbf{I}$ be a pocrim or BCK-algebra. Recall that if $\mathbf{I}$ is SI then its order reduct $\langle I; \leq \rangle$ has at most one atom, while if $\langle I; \leq \rangle$ has a unique atom then $\mathbf{I}$ is SI. Also, $\mathbf{I}$ is FSI iff 0 is meet irreducible in $\langle I; \leq \rangle$ (This is well known; [40, Proposition 6.5] contains a more general result.) Recall that an abelian group is FSI iff all of its subgroups are directly indecomposable, in which case its nonzero finitely generated subgroups are isomorphic either to $\mathbb{Z}$ or to $\mathbb{Z}_{p^n}$, where $0 < p, n \in \omega$ and p is prime. Up to isomorphism, the SI abelian groups are just the groups $\mathbb{Z}_{p^n}$, where $0 < n \in \omega \cup \{\infty\}$ and p is prime, while the simple abelian groups are just the groups $\mathbb{Z}_p$, where p is prime.

Proposition 12. *Let $\mathbf{A}$ be a sircomonoid or BCI-algebra and let $\mathbf{I}$ and $\mathbf{G}$ be its largest pocrim (or BCK-) subalgebra and its homomorphic image on $\{0 \dot{-} a : a \in A\}$, respectively. Assume that $\mathbf{A} \neq \mathbf{G}$.*

(i) *$\mathbf{A}$ is FSI iff for each nonzero $g \in G$, there exists a nonzero $i \in I$ such that $i \dot{-} g = -g$ (equivalently, in a sircomonoid, $g \oplus (-g) > 0$ for all nonzero $g \in G$) and $\mathbf{I}$ is FSI.*

(ii) *$\mathbf{A}$ is SI iff $\mathbf{A}$ is FSI and $\mathbf{I}$ is SI. In this case $\mathbf{A}$ and $\mathbf{I}$ have the same closed ideal as monolith.*

(iii) *If $\mathbf{A}$ is simple then $\mathbf{A} = \mathbf{I}$.*

Proof. (i) By relative 0-regularity, $\mathbf{A}$ and $\mathbf{I}$ are FSI iff $\{0\}$ is meet irreducible in their lattices of closed ideals. If K, L are closed ideals of $\mathbf{I}$ then, by the RCEP (Corollary 7), $\mathrm{Cig}^{\mathbf{A}}(K) \cap \mathrm{Cig}^{\mathbf{A}}(L) \cap I = K \cap L$. Since $I \neq \{0\}$, if $\mathbf{A}$ is FSI then so is $\mathbf{I}$. In this case if $0 \neq g \in G$ then some $j \in I \cap \mathrm{Cig}^{\mathbf{A}}(\{g\})$ is not 0, and $j \dot{-} (ng) \dot{-} (n(-g)) = 0$ for some positive $n \in \omega$. Choose the least such n and let $i = j \dot{-} ((n-1)g) \dot{-} ((n-1)(-g))$. Then $0 \neq i \in I$ and $i \dot{-} g = -g$. Conversely, if $\mathbf{I}$ is FSI but $K \cap L = \{0\}$ for nonzero closed ideals K, L of $\mathbf{A}$ then $K \cap I$ or $L \cap I$ is $\{0\}$, say $K \cap I = \{0\}$. Let $0 \neq k \in K$ and $g = 0 \dot{-} k$. Then $0 \neq g, -g \in G \cap K$ and if $0 \neq i \in I$ then $i \notin K$, whence $i \dot{-} g \neq -g$.

(ii) Let $\mathbf{A}$ be SI with monolith J. Since $I \neq \{0\}$, we have $J \subseteq I$ and J is clearly a closed ideal of $\mathbf{I}$. If K is a nonzero closed ideal of $\mathbf{I}$

then $J \subseteq \mathrm{Cig}^{\mathbf{A}}(K) \cap I = K$, by the RCEP, so $\mathbf{I}$ is SI with monolith J. Conversely, if $\mathbf{A}$ is FSI and $\mathbf{I}$ is SI with monolith J and K is a nonzero closed ideal of $\mathbf{A}$ then K contains $I \cap K$, which is a nonzero closed ideal both of $\mathbf{A}$ and of $\mathbf{I}$, hence $K \supseteq J$. Thus, $\mathbf{A}$ is SI with monolith J.

(iii) follows immediately from $\mathbf{G} \cong \mathbf{A}/I$.

The following construction shows that one can say little in general, beyond Lemma 3, about the order reduct of a SI sircomonoid (and still less for BCI-algebras) and that we can say nothing in general about their abelian group images.

Let $\mathbf{G} = \langle G; +, -, 0\rangle$ be any abelian group and $\mathcal{P} = \langle \mathbf{P}_g : g \in G\rangle$ a family of mutually disjoint posets. For each $g \in G$, let P_g be the universe of $\mathbf{P}_g$. Assume that each $g \in G$ is the least element of $\mathbf{P}_g$ and that $\mathbf{P}_0$ has a unique atom e and is the order reduct of a BCK-algebra $\mathbf{I}$ with $0 = 0^{\mathbf{I}}$. Let $A = \bigcup_{g \in G} P_g$ and let $\leq$ be the union of the partial orders of the posets $\mathbf{P}_g$, $g \in G$.

Proposition 13. *$\langle A; \leq\rangle$ is the order reduct of a SI BCI-algebra $\mathbf{A}$ (hence $\mathbf{I}$ is the largest BCK-subalgebra of $\mathbf{A}$) such that $\mathbf{G}$ is the unique group expansion of $\mathbf{A}$s homomorphic image on $\{0 \dot{-} a : a \in A\}$ and $\mathbf{CId}\,\mathbf{A}$ is isomorphic to the ordinal sum of $\mathbf{CId}\,\mathbf{I}$ and the subgroup lattice of $\mathbf{G}$. If in addition each poset $\mathbf{P}_g$ is bounded and $\mathbf{I}$ is a pocrim then we may choose $\mathbf{A}$ to be a sircomonoid with all the same properties.*

Proof. Consider first the case where $\mathbf{I}$ is a pocrim and each poset $\mathbf{P}_g$ has a greatest element t_g. We define $\oplus$ on A as follows. We require that 0 act as an identity element for $\oplus$ on A and that $\oplus$ extend the monoid operation of $\mathbf{I}$. If $g, h \in G$ are not both 0, and $0 \neq a \in P_g$ and $0 \neq b \in P_h$, we define $a \oplus b = t_{g+h}$. Then $\langle A; \oplus, 0; \leq\rangle$ is a commutative residuated pomonoid with minimal zero; its residuation $\dot{-}$ may be characterized as follows. For $a \in A$, $a \dot{-} 0 = a$ and if $a \leq b \in A$ then $a \dot{-} b = 0$. $\mathbf{I}$ is a subalgebra of the equivalent sircomonoid of $\langle A; \oplus, 0; \leq\rangle$. For $0 \neq g \in G$ and $a, b \in P_g$ with $a \nleq b$, we have $a \dot{-} b = e$. Finally, for distinct $g, h \in G$ and $a \in P_g$ and $0 \neq b \in P_h$, we have $a \dot{-} b = g - h$. If $\mathbf{I}$ is merely a BCK-algebra, the above characterization of $\dot{-}$ defines a binary operation on A, making $\langle A; \dot{-}, 0\rangle$ a BCI-algebra (even if the posets P_g are not bounded above). Since $e \dot{-} h = -h$ for all nonzero $h \in G$ and $\mathbf{I}$ is SI, $\mathbf{A}$ is SI with monolith $\mathrm{Cig}^{\mathbf{I}}(\{e\})$, by Proposition 12(i),(ii).[13] Moreover, every closed ideal K of $\mathbf{A}$ either contains or is contained in I. For if $a \in K \setminus I$ and $i \in I$ then $i \dot{-} a = 0 \dot{-} a \in G \cap K$, so $i \in K$. Then by the RCEP, the fact that $\mathbf{A}/I \cong \mathbf{G}$ and the (relative) Correspondence Theorem, $\mathbf{CId}\,\mathbf{A}$ is isomorphic to the ordinal sum of $\mathbf{CId}\,\mathbf{I}$ and $\mathbf{CId}\,\mathbf{G}$ (which is the subgroup lattice of $\mathbf{G}$).

3.3. Commutator Theory. Let K be a relatively congruence modular quasivariety and $\mathbf{A} \in \mathsf{K}$ and $\rho, \sigma \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$. For $\mu \in \mathrm{Con}\,\mathbf{A}$, let $Z(\rho, \sigma; \mu)$ signify that for all $\langle a, b\rangle \in \rho$ and all $\langle c, d\rangle \in \sigma$ and all binary polynomial operations $f(x, y)$ and $g(x, y)$ of $\mathbf{A}$ the following holds: if μ contains $\langle f(a, c), g(a, c)\rangle$, $\langle f(a, d), g(a, d)\rangle$ and $\langle f(b, c), g(b, c)\rangle$ then μ contains $\langle f(b, d), g(b, d)\rangle$ also. For $\mu \in \mathrm{Con}\,\mathbf{A}$, let $C(\rho, \sigma; \mu)$ signify that whenever t is a term operation of $\mathbf{A}$, if $t(\vec{a}^1, \vec{b}^1)\,\mu\,t(\vec{a}^1, \vec{b}^2)$ and $\vec{a}^1\,\rho\,\vec{a}^2$ and $\vec{b}^1\,\sigma\,\vec{b}^2$ (componentwise) then $t(\vec{a}^2, \vec{b}^1)\,\mu\,t(\vec{a}^2, \vec{b}^2)$.

The (absolute) *commutator* $[\rho, \sigma]$ of ρ and σ is the least congruence μ of $\mathbf{A}$ such that $C(\rho, \sigma; \mu)$ and $C(\sigma, \rho; \mu)$ [12, Definition 3.2]. The K-*commutator* $[\rho, \sigma]_{\mathsf{K}}$ of ρ and σ is the least K-congruence μ of $\mathbf{A}$ such that $Z(\rho, \sigma; \mu)$; it coincides with $[\sigma, \rho]_{\mathsf{K}}$ and, like $[\rho, \sigma]$, is contained in $\rho \cap \sigma$ and is isotone in both arguments [25, Theorem 2.13]. If μ is a K-congruence of $\mathbf{A}$ then $Z(\rho, \sigma; \mu)$ implies $C(\rho, \sigma; \mu)$ [25, Lemma 2.4], whence $[\rho, \sigma] \subseteq [\rho, \sigma]_{\mathsf{K}}$.

Lemma 14. [8] *Let K be an $\mathcal{L}$-quasivariety and 0 an $\mathcal{L}$-term that is constant over K, and assume that K is both relatively 0-regular and relatively congruence modular. Let $\mathbf{A} \in \mathsf{K}$ and $\rho, \sigma, \mu \in \mathbf{Con}_{\mathsf{K}}\,\mathbf{A}$. Then $Z(\rho, \sigma; \mu)$ iff $C(\rho, \sigma; \mu)$. Thus, $[\rho, \sigma]_{\mathsf{K}} = \Theta^{\mathbf{A}}_{\mathsf{K}}([\rho, \sigma])$.*

Proof. Let d_j, $j < m \in \omega$, be binary $\mathcal{L}$-terms such that (7) is true. Assume that $C(\rho, \sigma; \mu)$. Let $\langle a, b\rangle \in \rho$ and $\langle c, d\rangle \in \sigma$. Let $\vec{e}, \vec{f} \in A$,

---

[13] The special case (disregarding subdirect irreducibility) of this construction where P_g is a singleton for all nonzero $g \in G$ was noted for BCI-algebras in [18]. In this case $\dot{-}$ is determined by its restrictions to I and to G and subdirect irreducibility of $\mathbf{I}$ ensures that of $\mathbf{A}$, even if $\langle I; \leq\rangle$ has no atom.

$f(x,y) = r(x,y,\vec{e})$ and $g(x,y) = s(x,y,\vec{f})$, where $r(x,y,\vec{z}), s(x,y,\vec{w})$ are $\mathcal{L}$-terms. Suppose all of $\langle f(a,c), g(a,c)\rangle$, $\langle f(a,d), g(a,d)\rangle$, $\langle f(b,c), g(b,c)\rangle$ are contained in μ. Let $t(x,y,\vec{z},\vec{w}) = d_j(r(x,y,\vec{z}),\ s(x,y,\vec{w}))$, where $j < m$.

By (7), μ identifies each of $t(a,d,\vec{e},\vec{f}), t(a,c,\vec{e},\vec{f})$ with 0 and therefore with the other; similarly μ identifies $t(b,c,\vec{e},\vec{f})$ with 0. Since $C(\rho,\sigma;\mu)$, it follows that μ identifies $t(b,d,\vec{e},\vec{f})$ with $t(b,c,\vec{e},\vec{f})$ and therefore with 0, i.e., $\langle d_j(f(b,d),\ g(b,d)),\ 0\rangle \in \mu$. Now $j < m$ was arbitrary and μ is a K-congruence so, by (7), μ identifies $f(b,d)$ with $g(b,d)$. This shows that $Z(\rho,\sigma;\mu)$. Since the converse holds more generally, this establishes the first claim, from which the second follows by standard argument.

Corollary 15. *Let* K *be* SIRCOM *or* BCIA *and* $\mathbf{A} \in \mathsf{K}$ *with* $\rho, \sigma \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$. *Then* $0/[\rho,\sigma]_{\mathsf{K}} = 0/[\rho,\sigma]$.

Proof. Since K is relatively 0-regular and relatively congruence modular (Corollary 11), this follows from the previous lemma and Corollary 9.[14]

Let K be SIRCOM or BCIA. For $\mathbf{A} \in \mathsf{K}$ and $\rho, \sigma \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ with $0/\rho = J$ and $0/\sigma = M$, we may denote the common value of $0/[\rho,\sigma]$ and $0/[\rho,\sigma]_{\mathsf{K}}$ by $[J,M]$. (This is unambiguous, by the relative 0-regularity of K.) Note that $Z(\rho,\sigma;\mu)$ holds if $\mathbf{A}/\mu$ is termwise equivalent to an abelian group, since in that case the polynomials $f(x,y)$ and $g(x,y)$ have (modulo μ) the form $nx + my + p$ for some $n, m \in \mathbb{Z}$ and $p \in A$. It follows that $[J,M] \subseteq I$ for all closed ideals J, M of *any* $\mathbf{A} \in \mathsf{K}$, where $I = [0)$.

Theorem 16. *Let* K *be* SIRCOM *or* BCIA *and* $\mathbf{A} \in \mathsf{K}$. *Let* J, M *be closed ideals of* $\mathbf{A}$ *and let* $L = \{j \dot{-} (j \dot{-} m) \dot{-} (-(0 \dot{-} m)) : j \in J;\ m \in M\}$ *and* $I = [0)$. *Then* $L = [J,M] = J \cap M \cap I$. *Thus,* $[A,A] = I$ *and* $\mathbf{Con}_{\mathsf{K}}\,\mathbf{A}$ *satisfies the identity* $(\mathrm{C2})_{\mathsf{K}}$: $[x,y]_{\mathsf{K}} \approx x \cap y \cap [1,1]_{\mathsf{K}}$, *where* $1 = A^2$. *Moreover,* $[1,1]_{\mathsf{K}} = [1,1]$.

Proof. Take $j \in J = 0/\rho$ and $m \in M = 0/\sigma$, where $\rho, \sigma \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ and let $t(x_1,x_2,y) = x_1 \dot{-} (x_1 \dot{-} x_2) \dot{-} (y \dot{-} (y \dot{-} x_2))$. Then, we have

[14] In fact Corollary 15 is true of any relatively 0-regular quasivariety for which the statement of Corollary 10 is also true [8].

$C(\rho, \sigma; [\rho, \sigma]_{\mathsf{K}})$. Now in $\mathbf{A}$, $t(0, m, 0) = 0 = t(0, m, m)$, so

$$\langle j \dot{-} (j \dot{-} m) \dot{-} (-(0 \dot{-} m)), 0\rangle \;=\; \langle t(j, m, 0),\, t(j, m, m)\rangle \;\in [\rho, \sigma]_{\mathsf{K}},$$

i.e., $j \dot{-} (j \dot{-} m) \dot{-} (-(0 \dot{-} m)) \in [J, M]$. Thus, $L \subseteq [J, M] \subseteq J \cap M \cap I$. Conversely, if $a \in J \cap M \cap I$ then $0 \dot{-} a = 0$, whence

$$a \;=\; a \dot{-} 0 \dot{-} 0 \;=\; a \dot{-} (a \dot{-} a) \dot{-} (-(0 \dot{-} a)) \;\in L.$$

Let $\langle c, d\rangle \in [1, 1]_{\mathsf{K}}$. Then $c \dot{-} d, d \dot{-} c \in [A, A] = I$, so $-(0 \dot{-} c) \leq d \dot{-} (d \dot{-} c)$. If $s(x_1, x_2, y) = x_1 \dot{-} (y \dot{-} (y \dot{-} x_2))$ then $s(-(0 \dot{-} c), c, 0) = 0 = s(-(0 \dot{-} c), c, d)$ so $\langle c, d\rangle = \langle s(c, 0, 0), s(d, 0, d)\rangle \in [1, 1]$.

In particular, the abelian sircomonoids or BCI-algebras $\mathbf{A}$ (i.e., those with $[A, A] = \{0\}$) are affine[15], since $\mathbf{A}/[A, A]$ is *termwise* equivalent to an abelian group.[16] An algebra $\mathbf{A}$ in a quasivariety K is called K-*prime* (or *relatively prime*) if $[\rho, \sigma]_{\mathsf{K}}$ is a nonidentity K-congruence of $\mathbf{A}$ whenever both ρ and σ are. We call $\mathbf{A}$ *prime* if it is $\mathrm{V}(\{\mathbf{A}\})$-prime (i.e., if the foregoing condition holds with all Ks dropped). By the previous theorem, a FSI sircomonoid or BCI-algebra is affine or relatively prime. For congruence modular *varieties* V, the commutator identity (C2) is a consequence of the CEP alone[17] [27, Theorem 2.2] and is equivalent to the requirement that all SI members of V be affine or prime [26, Proposition 4.2].

3.4. Subvarieties. For a class M of similar algebras, we use $\mathrm{H}(\mathsf{M})$, $\mathrm{I}(\mathsf{M})$, $\mathrm{S}(\mathsf{M})$ and $\mathrm{P_U}(\mathsf{M})$ to denote, respectively, the classes of homomorphic images, of isomorphic images, of subalgebras and of ultraproducts

[15] An algebra $\mathbf{A}$ is *affine* if there is an abelian group $\langle A; +, -, 0\rangle$ with the same universe as $\mathbf{A}$ and a ternary term operation t of $\mathbf{A}$ such that for any $a, b, c \in A$, $t(a, b, c) = a - b + c$ and if $\mathbf{C} = \langle A; t\rangle$ then any n-ary term operation of $\mathbf{A}$ is a homomorphism from $\mathbf{C}^n$ to $\mathbf{C}$.

[16] Recall that the fundamental theorem of abelian algebras says that in a congruence modular *variety*, such algebras are always affine [16]. In a relatively modular quasivariety, however, an abelian algebra need not be affine [25, pp. 478–480]. The abelian members of SIRCOM and BCIA are characterized by their satisfaction of $x \dot{-} (x \dot{-} y) \approx y$; the latter have been called *p-semisimple BCI-algebras* in the literature.

[17] The (absolute) CEP fails even for pocrims and BCK-algebras, however [4]. Also, BCK-algebras need not be (absolutely) congruence modular [45] and it can be shown that the same is true of pocrims.

of members of M. If M is contained in a congruence modular variety then the prime algebras in $\mathrm{V}(\mathsf{M})$ are in $\mathrm{HSP_U}(\mathsf{M})$ [15]. It follows that if $\mathrm{V}(\mathsf{M})$ is a variety of sircomonoids or BCI-algebras then any SI algebra in $\mathrm{V}(\mathsf{M})$ is an abelian group (or reduct of such) or in $\mathrm{HSP_U}(\mathsf{M})$. Moreover, $\mathrm{HSP_U}(\mathsf{M}) = \mathrm{SHP_U}(\mathsf{M})$, by the CEP.

A quasivariety is called *n-finite* if every algebra in it that is generated by at most n elements is finite. Recall that a finitely generated variety [resp. quasivariety] is a class of the form $\mathrm{V}(\mathsf{M})$ [resp. $\mathrm{Q}(\mathsf{M})$] for some finite set M of finite algebras (which may be assumed a singleton without loss of generality), and that every such class is locally finite (i.e., it is n-finite for all $n \in \omega$).

We define $\langle \dot{-} \rangle$-terms $j_n(x, y)$ $(n \in \omega)$ as follows: $j_0(x, y) = x$;

$$j_{2n+1}(x, y) = y \dot{-} (y \dot{-} j_{2n}(x, y));\ j_{2n+2}(x, y) = x \dot{-} (x \dot{-} j_{2n+1}(x, y)).$$

By (2), SIRCOM satisfies $j_{n+1}(x, y) \leq j_n(x, y)$ for all n. By (4), it also satisfies

$$x \dot{-} (m+1)y \dot{-} (m+1)(0 \dot{-} y) \leq x \dot{-} my \dot{-} m(0 \dot{-} y) \text{for all } m.$$

Let K be SIRCOM or BCIA. Denote by K_n and $\mathsf{K}_{n,m}$ the class of all algebras in K satisfying $j_{n+1}(x, y) \approx j_n(x, y)$ and the class of all algebras in K_n satisfying $x \dot{-} (m + 1)y \dot{-} (m + 1)(0 \dot{-} y) \approx x \dot{-} my \dot{-} m(0 \dot{-} y)$, respectively.

Theorem 17. *Let* K *be* SIRCOM *or* BCIA.

(i) *The classes* K_n *and* $\mathsf{K}_{n,m}$ *are subvarieties of* K.

(ii) *Let* V *be a variety generated by a subclass* M *of* K *and* $\mathbf{F}$ *the* V*-free algebra on two free generators. If* $\langle F; \leq \rangle$ *satisfies the descending chain condition then* $\mathsf{V} \subseteq \mathsf{K}_{n,m}$ *for some* $n, m \in \omega$. *Thus:*

(iii) *Every 2-finite (e.g., every finitely generated) variety generated by a subclass of* K *is contained in* K *and is therefore congruence modular.*

(iv) *Every subquasivariety of* $\mathsf{BCIA}_{n,m}$ *is a variety. Thus, the* quasivariety *generated by any finite set of finite BCI-algebras is a congruence modular variety.*

Proof. (i) It suffices to note that (M2) and $j_{n+1}(x,y) \approx j_n(x,y)$ entail (M5).

(ii) Since $\mathbf{F}$ embeds in a direct product of members of M, $\mathbf{F} \in \mathsf{K}$, so the reference to $\leq$ makes sense. If $\overline{x}$ and $\overline{y}$ are the free generators of $\mathbf{F}$, the DCC forces $j_{n+1}(\overline{x},\overline{y}) = j_n(\overline{x},\overline{y})$ and $\overline{x} \dot{-} (m+1)\overline{y} \dot{-} (m+1)(0 \dot{-} \overline{y}) = \overline{x} \dot{-} m\overline{y} \dot{-} m(0 \dot{-} \overline{y})$ for some $n, m \in \omega$, whence $\mathsf{V} \subseteq \mathsf{K}_{n,m}$.

(iii) follows from Corollary 11 and the fact that finiteness implies the DCC.

(iv) If $\mathbf{A} \in \mathsf{BCIA}_{n,m}$ and $B = \{b_1, \ldots, b_r\} \subseteq A$ then the function $\varepsilon : A \to A$ defined by $\varepsilon(a) = a \dot{-} mb_1 \dot{-} \ldots \dot{-} mb_r \dot{-} m(0 \dot{-} b_1) \dot{-} \ldots \dot{-} m(0 \dot{-} b_r)$ is an endomorphism of $\mathbf{A}$: to verify this, modify the proof of [5, Lemma 4.3(ii)] in the obvious way. Also, $0/\ker\varepsilon = \mathrm{Cig}^{\mathbf{A}}(B)$, by Lemma 6. In view of (i), the rest of the proof is the same as that of [5, Theorem 4.4].

A subquasivariety of $\mathsf{SIRCOM}_{n,m}$ need not be a variety, even if it is finitely generated [5, pp. 300–301].

For a finitely generated relatively modular quasivariety K, there is always a finite bound on the size of its K-subdirectly irreducible algebras [25, Theorem 3.1]. It is an open problem (see [36], [25]) whether such a quasivariety is always finitely axiomatized by quasi-identities. A variety V is *residually small* if there is a cardinal bound $\kappa(\mathsf{V})$ on the size of its SI members; it has a *finite residual bound* if, moreover, $\kappa(\mathsf{V})$ may be chosen finite. A finitely generated congruence modular residually small variety has a finite residual bound and is finitely axiomatized by identities. If the congruence lattices of all algebras in a finitely generated congruence modular variety V satisfy (C2) then V is residually small [31], [12, Theorem 10.15]. Although SIRCOM and BCIA generate nonmodular varieties [45], these facts and the above theorem yield:

Corollary 18. *A variety generated by a finite sircomonoid has a finite residual bound and is finitely axiomatized by identities. A quasivariety generated a finite BCI-algebra is a finitely axiomatized variety with a finite residual bound.*

Theorem 19. SIRCOM [*resp.* BCIA] *has* $2^{\aleph_0}$ *subvarieties not generated by any class consisting of pocrims* [*resp. BCK-algebras*] *and abelian groups* [*resp. abelian group reducts*].

Proof. For each positive integer n, let $\mathbf{L}_n = \langle L_n; \leq \rangle$ be the $(2n+4)$-element poset depicted in Figure 2 below.

Note that $\mathbf{L}_n$ is a subposet of $\mathbf{L}_m$ only if $n = m$. Let $\mathbf{I}$ be the unique pocrim on the doubleton $I = \{0, 1\}$ (with $0 < 1$), and $\mathbf{G}$ the unique 2-element group with identity 0 on $G = \{0, g\}$. Assume that I is disjoint from each L_n. Let $\mathbf{A}_n$ be the SI sircomonoid with universe $I \cup L_n$ constructed as in the proof of Proposition 13 from $\mathbf{I}$, $\mathbf{G}$, $\mathbf{P}_0 = \langle I; \leq \rangle$ and $\mathbf{P}_g = \mathbf{L}_n$. Let $\mathsf{X} = \{\mathbf{A}_n : n \geq 1\}$ and for each n, let $\mathsf{X}_n = \{\mathbf{A}_m : m \neq n\}$. One may check that $\mathbf{A}_n \in \mathsf{SIRCOM}_2$,[18] so $\mathsf{V} = \mathrm{V}(\mathsf{X})$ is a congruence modular variety of sircomonoids with the CEP (see Theorem 17).

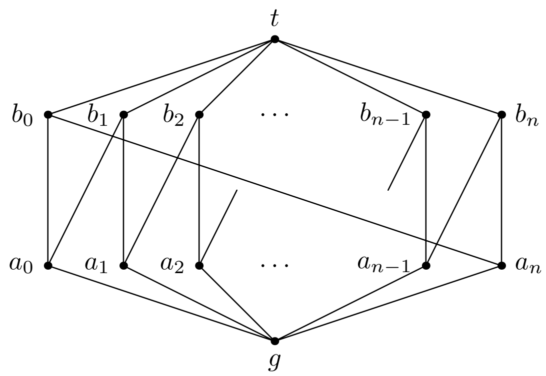

Figure 2.

For each n, since $\mathbf{A}_n$ is a finite algebra of finite type and SIRCOM is finitely axiomatized by quasi-identities, there is a universal first order sentence Φ_n over $\langle \oplus, \dot{-}, 0 \rangle$ such that the algebras of type $\langle 2, 2, 0 \rangle$ satisfying Φ_n are exactly the sircomonoids into which $\mathbf{A}_n$ is not embeddable. Thus, $\mathrm{ISP_U}(\mathsf{X}_n) \models \Phi_n$, and $\mathbf{A}_n \notin \mathrm{ISP_U}(\mathsf{X}_n)$.

By the remarks following Theorem 16, a SI algebra $\mathbf{B}$ in $\mathsf{V}_n = \mathrm{V}(\mathsf{X}_n)$ is an abelian group or lies in $\mathrm{SHP_U}(\mathsf{X}_n)$. The following information, common

[18] In fact, $\mathbf{A}_n \in \mathsf{SIRCOM}_{2,1}$.

to every $\mathbf{A}_m$, is expressible by a first order sentence and therefore persists in an ultraproduct $\mathbf{U} \in \mathrm{P_U}(\mathsf{X}_n)$: $\mathbf{I}$ is a subalgebra, being one of just two order-disjoint components of the algebra; the other component is bounded; the algebra is built from its components by the construction in the proof of Proposition 13. Thus, by Proposition 13, $\mathbf{U}$ is SI with monolith I and has no other proper nonzero closed ideal, so $\mathbf{G}$ is its only nontrivial proper homomorphic image. It follows that $\mathbf{B}$ is an abelian group or lies in $\mathrm{ISP_U}(\mathsf{X}_n)$, whence $\mathbf{B}$ is not $\mathbf{A}_n$. This means that $\mathbf{A}_n \notin \mathsf{V}_n$.

Consequently, distinct nonempty subsets of X generate distinct subvarieties of V, and there are $2^{\aleph_0}$ such subsets. Note that since an ultraproduct of pocrims and abelian groups is either a pocrim or an abelian group, none of these subvarieties is generated by the union of a class of pocrims and a class of abelian groups. This proves the result for SIRCOM; the argument for BCIA is similar.

3.5. Direct Decomposition Theorem. Let $\mathbf{A}$ be a sircomonoid or a BCI-algebra and let $\mathbf{I}$ and $\mathbf{G}$ be its largest pocrim (or BCK-) subalgebra and its homomorphic image on $\{0 \dot{-} a : a \in A\}$. Throughout this section, $+$ shall denote the binary operation on *all of* A defined by $a+b = a \dot{-} (0 \dot{-} b)$ $(a, b \in A)$. This extends the unique group operation of the abelian group expansion of $\mathbf{G}$. Observe that $\mathbf{A}$ satisfies $x + 0 \approx x$ and $(0 \dot{-} x) + x \approx 0$, as well as $(x + y) + z \leq x + (y + z)$. Indeed, over $\mathbf{A}$,

$$\begin{aligned}
&((x + y) + z) \dot{-} (x + (y + z)) \\
&\approx x \dot{-} (0 \dot{-} y) \dot{-} (0 \dot{-} z) \dot{-} (x \dot{-} (0 \dot{-} (y \dot{-} (0 \dot{-} z)))) \\
&\approx x \dot{-} (x \dot{-} (0 \dot{-} (y \dot{-} (0 \dot{-} z)))) \dot{-} (0 \dot{-} y) \dot{-} (0 \dot{-} z) \text{ (by (1))} \\
&\leq 0 \dot{-} (y \dot{-} (0 \dot{-} z)) \dot{-} (0 \dot{-} y) \dot{-} (0 \dot{-} z) \text{ (by (2))} \\
&\leq y \dot{-} (y \dot{-} (0 \dot{-} z)) \dot{-} (0 \dot{-} z) \text{ (by (M4))} \approx 0 \text{ (by (2))}.
\end{aligned}$$

Nevertheless, 0 may fail to be a left identity and $0 \dot{-} x$ a right inverse for $+$, and $+$ is not generally associative nor commutative on A.

Theorem 20. *Let* $\mathbf{A}$ *be a sircomonoid or BCI-algebra and* $\mathbf{I}$ *and* $\mathbf{G}$ *as above. The following conditions are equivalent.*

(i) **A** *satisfies the associative law* $(x+y)+z \approx x+(y+z)$*;*

(ii) **G** *acts on* **A** *by translation, i.e., for any* $a \in A$ *and* $g, h \in G$*,* $(a+g)+h = a+(g+h)$*;*

(iii) $\mathbf{A} \cong \mathbf{I} \times \mathbf{G}$*;*

(iv) G *is a closed ideal of* **A***;*

(v) *for each* $g \in G$*, the map* $a \mapsto a \dot{-} g$ *(*$a \in A$*) is a bijection of* A*;*

(vi) *for each* $g \in G$*, the map* $a \mapsto a \dot{-} g$ *(*$a \in A$*) is injective;*

(vii) *for each* $a \in A$ *and* $g \in G$*,* $a+g$ *is the largest* $x \in A$ *such that* $x \dot{-} g \leq a$*. [Note that for a sircomonoid* **A** (vii) *says that for each* $a \in A$ *and* $g \in G$*,* $a+g = a \oplus g$*.]*

Proof. First we prove the equivalence of the conditions when **A** is a BCI-algebra. (i) ⇒ (ii) is clear.

(ii) ⇒ (iii): By (ii), for every $a \in A$ and $g \in G$, we have

$$a = a \dot{-} (-g) \dot{-} g. \tag{9}$$

Indeed, by (M2) and (ii),

$$\begin{aligned} a &= a+0 = a+(g+(-g)) = (a+g)+(-g) \\ &= a \dot{-} (-g) \dot{-} (-(-g)) = a \dot{-} (-g) \dot{-} g. \end{aligned}$$

In particular, **A** satisfies

$$x \approx x \dot{-} (-(0 \dot{-} x)) \dot{-} (0 \dot{-} x),$$

since $0 \dot{-} a \in G$ for any $a \in A$. Now $\mathbf{G} \cong \mathbf{A}/I$ satisfies $x \approx -(0 \dot{-} x)$, so $a \dot{-} (-(0 \dot{-} a)) \in I$ for all $a \in A$. Thus, the map $\gamma : I \times G \to A$ defined by $\gamma(i,g) = i \dot{-} g$ is surjective.[19] Consider $i, j \in I$ and $g, h \in G$. We need to show that $i \dot{-} g \dot{-} (j \dot{-} h) = i \dot{-} j \dot{-} (g \dot{-} h)$. Now

$$\begin{aligned} & i \dot{-} j \dot{-} (g \dot{-} h) = (i \dot{-} j) + (-(g \dot{-} h)) \text{ (since } g \dot{-} h \in G) \\ &= (i \dot{-} j) + ((-g)+h) = ((i \dot{-} j)+(-g))+h \\ &= i \dot{-} j \dot{-} g \dot{-} (-h) = i \dot{-} g \dot{-} (-h) \dot{-} j \text{ (by (1))} \\ &= i \dot{-} g \dot{-} (-h) \dot{-} (j \dot{-} h \dot{-} (-h)) \text{ (by (9))} \\ &\leq i \dot{-} g \dot{-} (j \dot{-} h) \text{ (by (4))}. \end{aligned}$$

[19] The equivalence of (ii) and (iv) and the fact that (iv) implies surjectivity of γ were established (for BCI-algebras) in [19, Theorems 2, 4].

Conversely,

$$\begin{aligned} & i \dot{-} g \dot{-} (j \dot{-} h) = i \dot{-} g \dot{-} (-h) \dot{-} h \dot{-} (j \dot{-} h) \text{ (by (9))} \\ & \leq i \dot{-} g \dot{-} (-h) \dot{-} j \text{ (by (4))} \\ & = i \dot{-} j \dot{-} g \dot{-} (-h) \text{ (by (1))} \\ & = i \dot{-} j \dot{-} (g \dot{-} h) \text{ (by the previous calculation).} \end{aligned}$$

Thus, $\gamma : \mathbf{I} \times \mathbf{G} \to \mathbf{A}$ is a homomorphism. For $i \in I$ and $g \in G$, if $i \dot{-} g = 0$ then $i \leq g$ so $i = g \in I \cap G$, i.e., $i = g = 0$. This shows that γ is injective.

(iii) ⇒ (iv): By (iii), G is the 0-class of the congruence kernel of the canonical epimorphism $\mathbf{A} \to \mathbf{I} \in \mathsf{BCIA}$.

(iv) ⇔ (v) is proved in [20, Lemma 1.2] and (v) ⇒ (vi) is trivial.

(vi) ⇒ (vii): Let $a \in A$ and $g \in G$. By (1), (4) and (M2),

$$(a + g) \dot{-} g = a \dot{-} (-g) \dot{-} g = a \dot{-} g \dot{-} (0 \dot{-} g) \leq a \dot{-} 0 = a.$$

If $x \in A$ and $x \dot{-} g \leq a$ then

$$x \dot{-} (a + g) \dot{-} g = x \dot{-} g \dot{-} (a \dot{-} (-g)) \leq a \dot{-} (a \dot{-} (-g)) \leq -g \in G,$$

so $x \dot{-} (a + g) \dot{-} g = -g = 0 \dot{-} g$. By (vi), $x \dot{-} (a + g) = 0$, i.e., $x \leq a + g$.

(vii) ⇒ (i): Let $a, b, c \in A$. Then

$$\begin{aligned} & (a + (b + c)) \dot{-} (-(0 \dot{-} c)) = a \dot{-} (0 \dot{-} (b \dot{-} (0 \dot{-} c))) \dot{-} (-(0 \dot{-} c)) \\ & = a \dot{-} (-(0 \dot{-} c)) \dot{-} (0 \dot{-} b \dot{-} (-(0 \dot{-} c))) \text{ (by (1) and (5))} \\ & \leq a \dot{-} (0 \dot{-} b) \text{ (by (4))} = a + b. \end{aligned}$$

By (vii), since $-(0 \dot{-} c) \in G$, we have

$$a + (b + c) \leq (a + b) + (-(0 \dot{-} c)) = (a + b) \dot{-} (0 \dot{-} c) = (a + b) + c.$$

The reverse inequality is true in any BCI-algebra, so $+$ is associative on A.

Now let $\mathbf{A}$ be a sircomonoid. The only condition involving $\oplus$ is (iii) so by taking $\langle \dot{-}, 0 \rangle$-reducts, we deduce from the above that (iii) implies all of the other conditions and that these other conditions are equivalent. Conversely, let $\mathbf{A}$ satisfy (i), (ii) and (iv)-(vii). We must prove (iii). As above, the map $\gamma : I \times G \to A$ defined by $\gamma(i, g) = i \dot{-} g$ is a bijective $\langle \dot{-}, 0\rangle$-homomorphism. To show that it preserves $\oplus$, consider $i, j \in I$ and $g, h \in G$. Now $i \dot{-} g = i \dot{-} (-(-g)) = i + (-g) = i \oplus (-g)$ (by (vii)) and $j \dot{-} h = j \oplus (-h)$, so

$$\begin{aligned}(i \dot{-} g) \oplus (j \dot{-} h) &= i \oplus (-g) \oplus j \oplus (-h) = ((i \oplus j) + (-g)) + (-h) \\ &= (i \oplus j) \dot{-} g \dot{-} h = (i \oplus j) \dot{-} (g \oplus h),\end{aligned}$$

as required.

A *relative subvariety* of a quasivariety K is a subquasivariety of K that has the form $\mathsf{K} \cap \mathrm{V}(\mathsf{M})$ for some $\mathsf{M} \subseteq \mathsf{K}$. Let POCRIM, BCKA, AG and AGR denote, respectively, the classes of all pocrims, all BCK-algebras, all abelian groups and all subtraction reducts of abelian groups.

Corollary 21. *The quasivariety generated by* $\mathsf{POCRIM} \cup \mathsf{AG}$ [*resp. by* $\mathsf{BCKA} \cup \mathsf{AGR}$] *is a relative subvariety of* SIRCOM [*resp. of* BCIA]*; it is axiomatized by (M1)* [*resp. (M2)*] *– (M5) together with* $(x + y) + z \approx x + (y + z)$,[20] *where* $x + y = x \dot{-} (0 \dot{-} y)$.

Proof. This follows from Theorem 20 because quasivarieties are closed under direct products and because the above associative law is satisfied in all pocrims (where both sides are identically x) and in all abelian groups.

An algebra in a quasivariety K is called K-*disconnected* if it is the direct product of a K-congruence distributive algebra and an abelian algebra. If K is SIRCOM or BCIA, it follows that the relative subvarieties of K consisting of K-disconnected algebras are just those satisfying the associative law for $+$. If a congruence modular *variety* V is the (varietal) join of a congruence distributive variety D and an abelian variety A then every member of V is (V-) disconnected; in fact, a stronger result is proved in [16]. Although $\mathrm{V}(\mathsf{BCIA})$ is not congruence modular, the above corollary and Theorem 20 provide an analogous result. From this and the fact that there are $2^{\aleph_0}$

[20] or equivalently, with $(x + (0 \dot{-} y)) + (0 \dot{-} z) \approx x + ((0 \dot{-} y) + (0 \dot{-} z))$, in view of conditions (i) and (ii) of Theorem 20.

varieties of pocrims (in fact, Brouwerian semilattices) [28], [34] and $2^{\aleph_0}$ varieties of BCK- (in fact, Hilbert) algebras [43], it is not difficult to deduce that there are $2^{\aleph_0}$ subvarieties of $\mathrm{Q}(\mathsf{POCRIM} \cup \mathsf{AG})$ not consisting of pocrims only or of abelian groups only; and similarly for $\mathrm{Q}(\mathsf{BCKA} \cup \mathsf{AGR})$.

4. Connection with Linear Logic

There is an intimate connection between BCI-algebras and a deductive system **BCIP** extending the implication fragment of linear logic. Here we infer some properties of **BCIP** from the preceding algebraic results. First we explain the connection. For this purpose, we need no special properties of BCI-algebras beyond the fact that they form a relatively 0-regular quasivariety. The logic **BCIP** is algebraizable in the sense of the Blok-Pigozzi theory [2][21].

4.1. Assertional Logics

Let 0 be a (fixed) term of the language $\mathcal{L}$ that is constant over an $\mathcal{L}$-quasivariety K. The *assertional logic of* K is the (Hilbert-style) deductive system $\mathbf{S} = \mathbf{S}(\mathsf{K}) = \mathbf{S}(\mathsf{K}, 0)$ defined as follows. For a set $T' \cup \{u\}$ of $\mathcal{L}$-terms, $T', \vdash_{\mathbf{S}} u$ iff

$$\{t \approx 0 : t \in T'\} \models_{\mathsf{K}} u \approx 0. \qquad (*)$$

This notion was introduced by D. Pigozzi [37]. Because K is a quasivariety, $\mathbf{S}$ is indeed a ('finitary' and 'structural') deductive system in the usual sense (e.g., of [2]). In particular, $(*)$ is equivalent to the existence of some finite $T \subseteq T'$ such that

$$\mathsf{K} \models \big(\textstyle\bigwedge_{t \in T} t(\vec{x}) \approx 0\big) \Rightarrow u(\vec{x}) \approx 0.$$

In what follows, it is therefore a harmless notational convenience to assume, whenever an entailment $T \vdash u$ is under discussion, that $T \cup \{u\}$ is a *finite*

[21] This theory has wider application than we require here and this section can be read without prior knowledge of it. Where terminology from [2] appears, it may be considered locally defined by the context of its first occurrence (which is always in quotation marks) or disregarded without obscuring the results.

set of terms, all of the form $s(\vec{x})$, where $\vec{x}$ is an understood *finite* sequence of variables including all that occur in $T \cup \{u\}$.

By its definition, the consequence relation $\vdash_{\mathbf{S}}$ of $\mathbf{S}$ is interpretable in the equational consequence relation $\models_{\mathsf{K}}$ of K. Under our assumptions, K is relatively 0-regular exactly when it is the 'equivalent quasivariety semantics' of $\mathbf{S}$,[22] in which case $\mathbf{S}$ is 'algebraizable' [6, Theorem 5.2], [2].[23]

Suppose K is relatively 0-regular. Recall that there exist binary $\mathcal{L}$-terms d_j ($j < m \in \omega$), such that K satisfies $(\bigwedge_{j<m} d_j(x,y) \approx 0) \Leftrightarrow x \approx y$. Thus, there is an interpretation of $\models_{\mathsf{K}}$ in $\vdash_{\mathbf{S}}$ that is 'inverse' to the definition of $\mathbf{S}$:

$$\mathsf{K} \models \left(\bigwedge_{i<n} s_i(\vec{x}) \approx t_i(\vec{x})\right) \Rightarrow s(\vec{x}) \approx t(\vec{x})$$
$$\text{iff for all } k < m, \{d_j(s_i,t_i) : i < n,\ j < m\} \vdash_{\mathbf{S}} d_k(s,t)$$

Let $\mathbf{A} \in \mathsf{K}$. If $\xi \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ then $0^{\mathbf{A}}/\xi$ is closed under the consequence relation of $\mathbf{S}(\mathsf{K})$. This means that for any set $T \cup \{u\}$ of $\mathcal{L}$-terms for which $T \vdash_{\mathbf{S}(\mathsf{K})} u$, if $\vec{a} \in A$ and $t^{\mathbf{A}}(\vec{a}) \in 0^{\mathbf{A}}/\xi$ for all $t \in T$ then $u^{\mathbf{A}}(\vec{a}) \in 0^{\mathbf{A}}/\xi$. Subsets of A with this closure property are called $\mathbf{S}(\mathsf{K})$-*filters* of $\mathbf{A}$. In fact the $\mathbf{S}(\mathsf{K})$-*filters* of $\mathbf{A}$ are *just* the sets $0^{\mathbf{A}}/\xi$, $\xi \in \mathrm{Con}_{\mathsf{K}}\,\mathbf{A}$ [6, Theorem 5.2].[24] As we observed earlier, these form an algebraic lattice when ordered by inclusion; we denote by $\mathrm{Fg}^{\mathbf{A}}_{\mathsf{K}}(Y)$ the least $\mathbf{S}(\mathsf{K})$-filter of $\mathbf{A}$ containing $Y \subseteq A$.

The assertional logics of relative subvarieties of a relatively 0-regular quasivariety K are (algebraizable) *axiomatic* extensions of $\mathbf{S}(\mathsf{K})$, i.e., they may be axiomatized by the union of an axiomatization of $\mathbf{S}(\mathsf{K})$ and a set of rules of the form $\emptyset \vdash u$; conversely all axiomatic simple extensions of $\mathbf{S}(\mathsf{K})$ arise in this way (and are algebraizable). Here an extension of a logic $\mathbf{S}$ is called *simple* if it has the same language as $\mathbf{S}$.

A deductive system $\mathbf{S}$ over $\mathcal{L}$ is said to have a *local deduction detachment theorem* (LDDT) if there is a family $\mathcal{E} = \{E_i(p,q) : i \in Y\}$ of finite

[22] Within our framework and for our purposes, this fact may serve as a local definition.

[23] Algebraizability of $\mathbf{S}(\mathsf{K})$ doesn't force K to be relatively 0-regular but this needn't concern us here.

[24] Here one cannot drop the assumption that K is relatively 0-regular, although it can be weakened: see [6, Proposition 6.1, Example 6.1], [8].

sets $E_i = E_i(p,q) = \{e_{ij}(p,q) : j \leq n_i \in \omega\}$ of binary $\mathcal{L}$-terms e_{ij} such that for any set $T \cup \{r,s\}$ of $\mathcal{L}$-terms, the following is true:

$$T, r \vdash_{\mathbf{S}} s \text{ iff there exists } i \in Y \text{ such that for all } j < n_i,\ T \vdash_{\mathbf{S}} e_{ij}(r,s).$$

In this case, $\mathcal{E}$ is called a *local deduction detachment system* for **S**. If in addition, we may choose $|\,Y\,| = 1$, say $\mathcal{E} = \{E\}$, we call E a *deduction detachment set* for **S** and say that **S** has a *deduction detachment theorem* (DDT).

The assertional logic $\mathbf{S} = \mathbf{S}(\mathsf{K})$ of a relatively 0-regular quasivariety K has a LDDT if and only if K has the RCEP [1]. In this case, $\mathcal{E}$ (as above) is a local deduction detachment system for $\mathbf{S} = \mathbf{S}(\mathsf{K})$ just when, for every $\mathbf{A} \in \mathsf{K}$ and $a, b \in A$, we have

$$b \in \mathrm{Fg}^{\mathbf{A}}_{\mathsf{K}}(\{a\}) \text{ iff there exists } i \in Y \text{ such that for all } j < n_i,\ e^{\mathbf{A}}_{ij}(a,b) = 0^{\mathbf{A}}$$

(see [6, Proposition 8.1]).

4.2. The logic BCIP. Since BCIA is relatively 0-regular, it is the equivalent quasivariety semantics of its (algebraizable) assertional logic $\mathbf{S}(\mathsf{BCIA})$, and $x \dot{-} y$ and $y \dot{-} x$ play the roles of the $d_j(x,y)$, $j < m$, above. In order that $\mathbf{S}(\mathsf{BCIA})$ have a more familiar logical appearance, let us replace BCIA by a quasivariety $\mathsf{BCIA}^{\rightarrow}$ with language $\langle \rightarrow \rangle$ to which it is termwise definitionally equivalent: in the axiomatization we replace all expressions of the form $s \dot{-} t$ by $t \rightarrow s$ and all occurrences of 0 by $x \rightarrow x$, for a variable x not occurring in the axiom. (M3) becomes $x \rightarrow x \approx y \rightarrow y$ so, over $\mathsf{BCIA}^{\rightarrow}$, $x \rightarrow x$ defines a constant term c. Henceforth, by $\mathbf{S}(\mathsf{BCIA})$ we really mean $\mathbf{S}(\mathsf{BCIA}^{\rightarrow}, \mathsf{c})$ and by a BCI-algebra a member of $\mathsf{BCIA}^{\rightarrow}$, but we shall not labour this distinction by enforcing these further notational changes. By $(*)$, the following theorems and rules are derivable in $\mathbf{S}(\mathsf{BCIA})$:

(B) $\vdash (p \rightarrow q) \rightarrow ((r \rightarrow p) \rightarrow (r \rightarrow q))$,
(C) $\vdash (p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$,
(I) $\vdash p \rightarrow p$,

(P) $\vdash (p \to p) \to (q \to q)$,
(MP) $p, p \to q \vdash q$,
(G) $p, q \vdash p \to q$,
(H) $p \vdash p \to (q \to q)$.

The rule (G) is sometimes called the *Gödel Rule*. Kabziński [23] pointed out that (B), (C), (I), (MP) and (G) axiomatize $\mathbf{S}(\mathsf{BCIA})$. The logic $\mathbf{S}(\mathsf{BCIA})$ has a decidable set of theorems [24]. The purely implicational deductive system axiomatized by (B), (C), (I) and (MP) is just the implication fragment of Girard's linear logic and is known as **BCI**; it was originally considered in its own right by Meredith. It is known that **BCI** is not algebraizable[25] [2, Theorem 5.9]. Thus, as observed by Kabziński [23], $\mathbf{S}(\mathsf{BCIA})$ is a proper extension of **BCI** and BCI-algebras do not constitute an algebraic semantics for **BCI**.

Consider a simple extension **S** of **BCI**. We shall say that **S** is *pointedly algebraizable* if $\mathbf{S} = \mathbf{S}(\mathsf{K}, \mathsf{c})$ for a quasivariety K and term c such that $\mathsf{K} \models x \to x \approx y \to y \approx \mathsf{c}$.[26] The next result clarifies the status of $\mathbf{S}(\mathsf{BCIA})$ among such extensions.

Proposition 22. $\mathbf{S}(\mathsf{BCIA})$ *is the least pointedly algebraizable simple extension of* **BCI** *and is a purely axiomatic extension of* **BCI**. *In the presence of (B), (C), (I) and (MP), it is axiomatized by any one of (P), (G) or (H).*

Proof. Let $\mathbf{S} = \mathbf{S}(\mathsf{K})$ be a pointedly algebraizable simple extension of **BCI**. By pointed algebraizability, $\mathsf{K} \models (x \approx \mathsf{c} \text{ and } y \approx \mathsf{c}) \;\Rightarrow\; x \to y \approx \mathsf{c}$. Then (G) is an **S**-derivable rule, by $(*)$, so **S** extends $\mathbf{S}(\mathsf{BCIA})$ (by Kabzinski's axiomatization), proving the first assertion. We know that (H) is an $\mathbf{S}(\mathsf{BCIA})$-derivable rule. Now $p \to p \vdash (p \to p) \to (q \to q)$ is an instance of (H) so, using (I), we infer that (P) is derivable in the

[25] **BCI** is 'equivalential' in the sense of [38]. From this fact and [2, Theorem 4.7], it follows that the extension of **BCI** by $p \vdash p \to (p \to p)$ is algebraizable.

[26] This does force **S** to be algebraizable: see the next proof. Not every algebraizable simple extension of **BCI** is pointedly algebraizable. **BCI**'s extension by the 'mingle' axiom $\vdash p \to (p \to p)$ is such an exception; it is a subsystem of the relevance logic **RM**.

extension of **BCI** by (H). Finally, in the extension $\mathbf{S}'$ of **BCI** by (P), one may derive $\vdash q \to ((p \to p) \to q)$ (apply (C) and (MP) to (P)) and therefore $p \vdash (p \to p) \to p$. Then by [2, Theorems 4.7, 2.17] and (P), $\mathbf{S}' = \mathbf{S}(\mathsf{K}', \mathsf{c})$ for some relatively c-regular $\langle \to \rangle$-quasivariety K' over which c is definable as $x \to x$. As above, it follows that (G) is derivable in $\mathbf{S}'$. Thus, any one of (P), (G) or (H) extends **BCI** to $\mathbf{S}(\mathsf{BCIA})$[27] and this is an axiomatic extension, in view of (P).

Because of this result, we denote $\mathbf{S}(\mathsf{BCIA})$ as **BCIP** henceforth. The **BCIP**-filters of BCI-algebras $\mathbf{A}$ are just their closed ideals, so the operators $\mathrm{Fg}^{\mathbf{A}}_{\mathsf{BCIA}}$ and $\mathrm{Cig}^{\mathbf{A}}_{\mathsf{BCIA}}$ coincide on subsets of A. Corollary 10 may be rephrased as:

Proposition 23. *The quasivariety of BCI-algebras and the variety generated by it have the same assertional logic, viz.,* **BCIP**.

We use the following abbreviations: $p \to^0 q$ means q; for $n \in \omega$, $p \to^{n+1} q$ abbreviates $p \to (p \to^n q)$. The discussion following Corollary 7 yields:

Proposition 24. **BCIP** *has a local deduction detachment theorem with local deduction detachment system* $\mathcal{E} = \{E_{nm} : n, m \in \omega\}$, *where* $E_{nm} = \{p \to^n ((p \to (p \to p)) \to^m q)\}$. *In other words,* $T, r \vdash_{\mathbf{BCIP}} s$ *iff for some* $n, m \in \omega$, $T \vdash_{\mathbf{BCIP}} r \to^n ((r \to (r \to r)) \to^m s)$.

This contrasts with the well known fact that **BCI** does not have a local deduction detachment theorem. It also unifies the results [1] that the assertional logics of BCK-algebras and of abelian groups each have a LDDT. The assertional logic **BCK** of BCKA is the (proper) axiomatic extension of **BCI** (or of **BCIP**) by the axiom (called the rule of *weakening*)

(K) $\vdash p \to (q \to p)$.

In order that the assertional logic of a 0-regular quasivariety K have a (full) deduction detachment theorem, K must be relatively congruence distributive [1], [3]. If $\mathbf{A}$ belongs to a relative subvariety K of BCIA and $\mathbf{G}$ is $\mathbf{A}$'s

[27] It follows that we could have adopted a simpler definition: a simple extension of **BCI** is pointedly algebraizable if and only if it has (P) as a theorem.

retract on $\{0 \dot{-} a : a \in A\}$ then $\mathbf{G}$ is termwise equivalent to an abelian group and therefore generates a subvariety of K that is not congruence distributive, unless $\mathbf{G}$ is trivial. Thus, a relative subvariety of BCIA whose assertional logic has a DDT must consist of BCK-algebras and these relative subvarieties of BCKA have been characterized [5, Theorem 4.2]. In logical terms, this amounts to:

Proposition 25. *The axiomatic extensions of* **BCIP** *with a deduction detachment theorem are just the axiomatic extensions of* **BCK** *in which, for some* $n \in \omega$, *the expression* $(p \rightarrow^{n+1} q) \rightarrow (p \rightarrow^{n} q)$ *is a theorem.*

Our discussion can be enlarged to include the (algebraizable) assertional logic of sircomonoids, which also extends a non-algebraizable fragment of linear logic; $\oplus$ represents the logical 'fusion of premisses' connective.[28] If the order reduct of a BCI-algebra $\mathbf{A}$ is a semilattice or bounded above or below, however, then $\mathbf{A}$ is a BCK-algebra. Consequently, any logic that extends both **BCIP** and a fragment of linear logic with $\wedge$ or $\vee$ already extends **BCK**.

Acknowledgement. Both authors thank the Japan Advanced Institute of Science and Technology and the South African National Research Foundation for financial support, and Professors Hiroakira Ono and Tomasz Kowalski for their hospitality in November 1999.

References

[1] W.J. Blok, D. Pigozzi, *Local deduction theorems in algebraic logic*, Colloquia Mathematica Societatis János Bolyai 54. Algebraic Logic, Budapest (Hungary), 1988, 75–109.

[2] W.J. Blok, D. Pigozzi, "Algebraizable Logics", Memoirs of the American Mathematical Society, Number 396, Amer. Math. Soc., Providence, 1989.

[28] See [39] for an analogous and more detailed discussion.

[3] W.J. Blok, D. Pigozzi, *Abstract Algebraic Logic and the Deduction Theorem*, Bulletin of Symbolic Logic, in print.

[4] W.J. Blok, J.G. Raftery, *Failure of the congruence extension property in BCK-algebras and related structures*, Math. Japonica **38** (1993), 633–638.

[5] W.J. Blok, J.G. Raftery, *Varieties of commutative residuated integral pomonoids and their residuation subreducts*, J. Algebra, **190** (1997), 280–328.

[6] W.J. Blok, J.G. Raftery, *Ideals in quasivarieties of algebras*, "Models, Algebras and Proofs" (X. Caicedo and C.H. Montenegro, eds.), Lecture Notes in Pure and Applied Mathematics, Vol. 203, Marcel Dekker, New York, 1999, pp. 167–186.

[7] W.J. Blok, J.G. Raftery, *On congruence modularity in varieties of logic*, Algebra Universalis, in print.

[8] W.J. Blok, J.G. Raftery, *On assertional logic*, manuscript.

[9] K. Fichtner, *Varieties of universal algebras with ideals* (Russian), Mat. Sbornik **75** (**117**) (1968), 445–453.

[10] K. Fichtner, *Eine Bemerkung über Mannigfaltigkeiten universeller Algebren mit Idealen*, Monatsch. d. deutsch. Akad. d. Wiss. (Berlin) **12** (1970), 21–25.

[11] I. Fleischer, *Every BCK-algebra is a set of residuables in an integral pomonoid*, J. Algebra **119** (1988), 360–365.

[12] R. Freese, R. McKenzie, "Commutator Theory for Congruence Modular Varieties", London Math. Soc. Lecture Note Ser., No. 125, Cambridge Univ. Press, 1987.

[13] H.-P. Gumm, A. Ursini, *Ideals in universal algebras*, Algebra Universalis **19** (1984), 45–54.

[14] J. Hagemann, *On regular and weakly regular congruences*, Preprint No. 75, Technische Hochschule Darmstadt, June 1973.

[15] J. Hagemann, C. Herrmann, *A concrete ideal multiplication for algebraic systems and its relation to congruence distributivity*, Arch. Math. (Basel) **32** (1979), 234–245.

[16] C. Herrmann, *Affine algebras in congruence modular varieties*, Acta Sci. Math. (Szeged) **41** (1979), 119–125.

[17] D. Higgs, *Dually residuated commutative monoids with identity element as least element do not form an equational class*, Math. Japonica **29** (1984), 69–75.

[18] C.S. Hoo, *Closed ideals and p-semisimple BCI-algebras*, Math. Japonica **35** (1990), 1103–1112.

[19] W. Huang, *On the p-semisimple part in BCI-algebras*, Math. Japonica **37** (1992), 159–161.

[20] W. Huang, *On BCI-algebras and semigroups*, Math. Japonica **42** (1995), 59–64.

[21] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad. **42** (1966), 26–29.

[22] K. Iséki, *On BCK-algebras with condition* (S), Math. Japonica **24** (1980), 625–626.

[23] J.K. Kabziński, *BCI-algebras from the point of view of logic*, Bull. Sect. Logic, Polish Acad. Sci., Inst. Philos. and Socio. **12** (1983), 126–129.

[24] R. Kashima, Y. Komori, *The word problem for free BCI-algebras is decidable*, Math. Japonica **37** (1992), 1025–1029.

[25] K. Kearnes, R. McKenzie, *Commutator theory for relatively modular quasivarieties*, Trans. Amer. Math. Soc. **331** (1992), 465–502.

[26] E.W. Kiss, *Injectivity in modular varieties I. Two commutator properties*, Bull. Austral. Math. Soc. **32** (1985), 33–44.

[27] E.W. Kiss, *Injectivity in modular varieties II. The congruence extension property*, Bull. Austral. Math. Soc. **32** (1985), 45–53.

[28] P. Köhler, *Brouwerian semilattices*, Trans. Amer. Math. Soc. **268** (1981), 103–126.

[29] M. Kondo, *Congruences and closed ideals in BCI-algebras*, Math. Japonica **46** (1997), 491–496.

[30] T. Lei, C. Xi, *P-radical in BCI-algebras*, Math. Japonica **30** (1985), 511–517.

[31] R.N. McKenzie, *Finite equational bases for congruence modular varieties*, Algebra Universalis **24** (1987), 224–250.

[32] J. Meng, X.L. Xin, *Characterization of atoms in BCI-algebras*, Math. Japonica **37** (1992), 359–361.

[33] H. Ono, Y. Komori, *Logics without the contraction rule*, J. Symbolic Logic **50** (1985), 169–202.

[34] T.M. Owens, "Varieties of Skolem Rings", PhD Thesis, Purdue University, 1974.

[35] M. Pałasinski, *An embedding theorem for BCK-algebras*, Math. Seminar Notes Kobe Univ. **10** (1982), 749–751.

[36] D. Pigozzi, *Finite basis theorems for relatively congruence distributive quasivarieties*, Trans. Amer. Math. Soc. **310** (1988), 499–533.

[37] D. Pigozzi, *Fregean algebraic logic*, Colloquia Mathematica Societatis János Bolyai 54. Algebraic Logic, Budapest (Hungary), 1988, 473–502.

[38] T. Prucnal, A. Wroński, *An algebraic characterization of the notion of structural completeness*, Bull. Sect. Logic, Polish Acad. Sci., Inst. Philos. and Socio. **3** (1974), 30–33.

[39] J.G. Raftery, C.J. van Alten, *On the algebra of noncommutative residuation: polrims and left residuation algebras*, Math. Japonica **46** (1997), 29–46.

[40] C.J. van Alten, J.G. Raftery, *On quasivariety semantics of fragments of intuitionistic propositional logic without exchange and contraction rules*, Reports on Mathematical Logic **31** (1997), 3–55; also Internal Report No. 5/99, School of Mathematical and Statistical Sciences, University of Natal, Durban.

[41] C.J. van Alten, J.G. Raftery, *On the lattice of varieties of residuation algebras*, Algebra Universalis, in print.

[42] S.M. Wei, Y.B. Jun, *Ideal lattices of BCI-algebras*, Math. Japonica **44** (1996), 303–305.

[43] A. Wroński, *The degree of completeness of some fragments of intuitionistic propositional logic*, Reports on Mathematical Logic **2** (1974), 55–62.

[44] A. Wroński, *BCK-algebras do not form a variety*, Math. Japonica **28** (1983), 211–213.

[45] A. Wroński, *Reflections and distensions of BCK–algebras*, Math. Japonica **28** (1983), 215–225.

School of Mathematical and Statistical Sciences,
University of Natal, King George V Avenue,
Durban 4001, South Africa
email: `raftery@scifs1.und.ac.za`

Department of Mathematics, Statistics and Computer Science,
University of Illinois at Chicago, 851 S Morgan Street,
Chicago, Illinois 60607-7045, USA
email: `cjvanalten@yahoo.com`