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"Terrible Twos"

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Chapter 73

A260016

The Terrible Twos Problem

"No definition of science is complete without a reference to terror."
Don DeLillo, *Ratner's Star*

On one cool April day a few years ago in New York City, I approached a street vendor in order to purchase a pretzel. While waiting, I observed the following enigmatic encryption written in chalk on the dirty street: $5 = 2^2 + 1$. We will probably never know who wrote this and why it was written, but the equation stimulated me to conduct the "Terrible Twos" contest in August of 1991. In this contest, participants were to construct numbers using just ones and twos, and any number of $+$, $-$, and $\times$ signs. People were also allowed exponentiation. As an example, let's first consider the problem where only the digit one is allowed. The number 80 could be written:

$$80 = (1 + 1 + 1 + 1 + 1) \times (1 + 1 + 1 + 1) \times (1 + 1 + 1 + 1) \qquad (73.1)$$

If we let $f(n)$ be the least number of digits that can be used to represent n, then we see that $f(80) \leq 13$. A contest which allows only ones for forming small numbers turns out not to be very interesting. However, once the digit 2 is also allowed, the problem becomes fascinating. Here is an example:

$$81 = \left(2^{2+1} + 1\right)^2 \qquad (73.2)$$

Here $f(81) \leq 5$. Is this the best you can do?

The explicit goal of The Terrible Twos Contest was to represent the numbers 20, 120, and 567 with as few digits as possible. I received hundreds of responses, and wish that I could report all of the observations and entries in this chapter. Here are some examples. The first triplet of answers came from R. Lankinen of Helsinki, Finland:

$$f(20) \leq 5, \text{ for } 20 = 2^{2+2} + 2 + 2 \qquad (73.3)$$

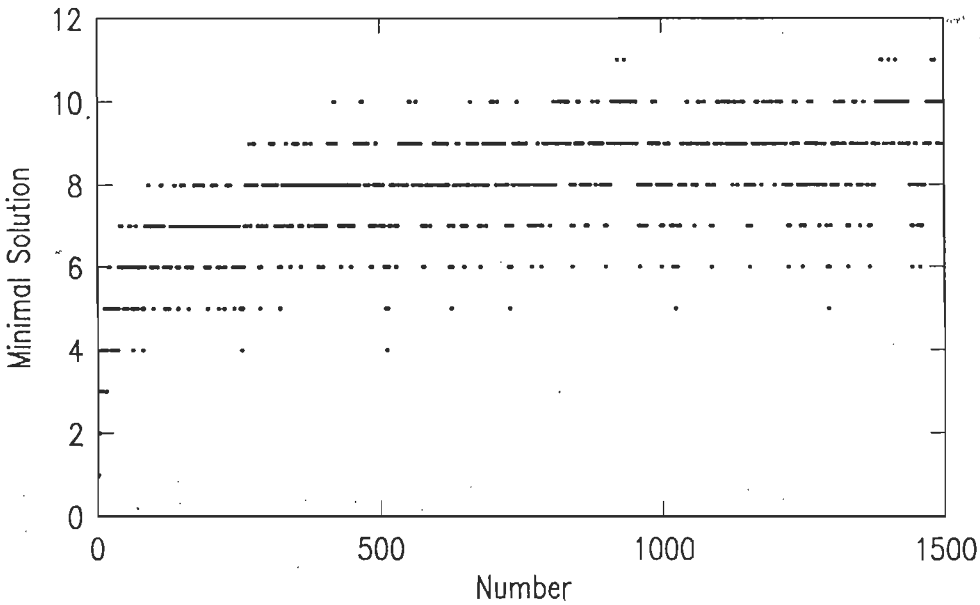

Figure 73.1. *Minimal integer solutions.* These solutions $f(n)$ were found for the first 1500 numbers. Concatenation of integers is not allowed.

$$f(120) \leq 6, \text{ for } 120 = ((2+1)^2+2)^2 - 1 \tag{73.4}$$

$$f(567) \leq 9, \text{ for } 567 = 2 \times 2 \times ((2 \times (2 \times 2 + 2))^2 - 2) - 1 \tag{73.5}$$

But is this the best one can do for the three numbers? It turns out that 567 can be constructed with just 8 digits. In fact, the contest winner, who first computed the minimum values for all three numbers, is Dan Hoey of Washington DC. Here are his minimal answers (which, I believe, use the smallest possible number of digits):

$$f(20) \leq 5 \text{ for } 20 = (1+2+2) \times (2+2) \tag{73.6}$$

$$f(120) \leq 6 \text{ for } (2 + (1+2)^2)^2 - 1 \tag{73.7}$$

$$f(567) \leq 8 \text{ for } (2^{2+2+2} - 1) \times (2+1)^2 \tag{73.8}$$

The contest becomes more interesting if we allow concatenation of digits (thus permitting multidigit numbers such as 11, 12, 121, etc.) For this case, the winning entries come from Mark McKinzie of the University of Wisconsin's Mathematics Department. Here are Mark's answers:

$$f(20) \leq 3 \text{ for } 20 = 22 - 2 \tag{73.9}$$

$$f(120) \leq 4 \text{ for } 120 = 11^2 - 1 \tag{73.10}$$

$$f(567) \leq 6 \text{ for } 567 = (2+1)^{2+1} \times 21 \tag{73.11}$$

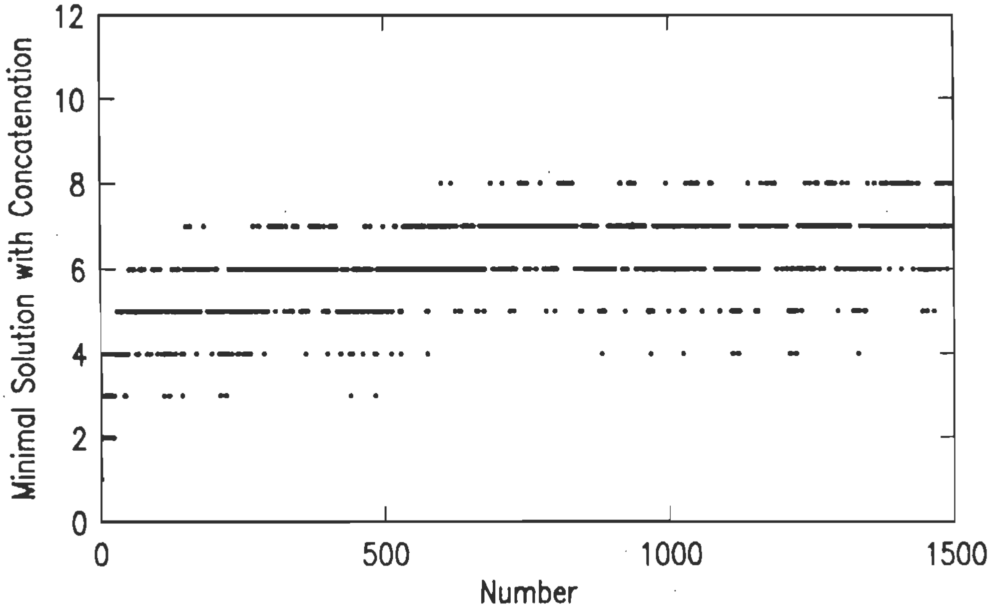

Figure 73.2. *Minimal integer solutions with concatenation.* The solutions $f(n)$ were found for the first 1500 numbers. Concatenation of integers is allowed (that is, multidigit numbers such as 12 and 121 are permitted).

Another equally successful set of answers comes from Ya-xiang, Beijing, China:

$$f(20) \leq 3 \text{ for } 20 = 21 - 1 \quad (73.12)$$

$$f(120) \leq 4 \text{ for } 120 = 121 - 1 \quad (73.13)$$

$$f(567) \leq 6 \text{ for } 21 \times (2 + 1)^{2+1} \quad (73.14)$$

Other minimal answers were submitted, and I must confess that the winners of the contest were sometimes determined as much by the speeds of our electronic communication networks as by intellectual prowess.

I collaborated with Ken Shirriff of the University of California for much of the analysis of this problem. Ken wrote a computer program in C which not only searches for the minimal solution for the first 1500 integers but also searches for the number of minimal ways to construct a number. For example, without allowing concatenation (multidigit numbers), he finds that there are 208 different ways to write the number 20, and 1128 different ways to write the number 21! Even more exciting is the fact that these 208 and 1128 different ways to write minimal solutions change to just 2 ways and 1 way if concatenation is allowed. (After all, there is just one way to minimally write 21 by concatenating 2 and 1.)

The program finds solutions by using dynamic programming techniques. It starts with the one digit base cases, and combines these numbers to generate all numbers that have shortest solution of two digits. The one and two digit results are combined to yield all numbers with three digit shortest solutions. This process continues until all the desired numbers have been found. In order to keep the computations from growing too quickly, Ken Shirriff prunes the results by dis-

Without Multidigit Expressions:		With multidigit expressions:	
Digits	Hard Number	Digits	Hard Number
2	3	2	3
3	2	3	5
4	7	4	7
5	13	5	29
6	21	6	51
7	41	7	151
8	91	8	601
9	269	9	1631
10	419	10	7159
11	921	11	19145
12	2983	12	71515
13	8519	13	378701
14	18859		
15	53611		
16	136631		
17	436341		

Figure 73.3. *Hard numbers.*

carding any results over 10000. He also limits results to integers by only using positive exponents. While the first limit probably has no effect on the results, there are a handful of shorter solutions that are only obtained by using negative exponents.

Figure 73.1 and Figure 73.2 show plots of our computed values of $f(n)$ vs. n for both non-concatenation and concatenation contests. Interestingly, minimal solutions comprised of less than 12 digits can be found for all numbers tested (on average, one needs about 7 digits to minimally construct n, $1 \leq n \leq 1500$).

73.1 Hard Numbers

"He could find how numbers behaved, but he could not explain why. It was his pleasure to hack his way through the arithmetical jungle, and sometimes he discovered wonders that more skillful explorers had missed."

Arthur C. Clarke, 1956, *The City and The Stars*

Let us also define the concept of "hard numbers" $f_h(n)$ which are the smallest numbers requiring $f(n)$ digits. For example, 921 is the smallest number which requires a walloping 11 digits for its expression. Running his program on the integers up to one million, Shirriff found the hard numbers listed in Figure 73.3. Plots of n vs. $f_h(n)$ seem to increase exponentially.

73.2 Unusual Solutions

The contest winner, Dan Hoey, also wrote a Lisp program to confirm his hand calculations, and as with Shirriff's C program, he did not initially check for negative exponents. However, he later extended his program to negative exponents, and discovered they sometimes result in shorter solutions. For instance, Hoey notes that if negative exponents are not checked, one might conclude that $f(640) = 8$. However, look at Hoey's amazing solution $f(640) = 7$ found when using negative exponents:

$$640 = (2^{((2+1)^2)}) \times (1 + 2^{-2}). \tag{73.15}$$

Nevertheless, he believes that 20, 120, and 567 do not benefit from the use of negative exponents unless some subexpression has a denominator or numerator exceeding 10^{12}. He found an interesting solution with negative exponents for 567:

$$567 = (2^{2^2} + 2)^2 \times (2 - 2^{-2}) \tag{73.16}$$

He further wonders whether future searches should consider using irrational numbers. Hoey writes, "In the same way that negative exponents imply fractions, fractional exponents imply irrational numbers, and then irrational exponents imply transcendental numbers. In fact, one could obtain complex numbers, too, but I don't think that is any help, and you have problems with branch cuts there." One question is whether there are any "integers" that benefit (in the sense of requiring fewer ones and twos) by considering and using irrational numbers, or rational numbers formed with fractional exponents. Is there any integer that benefits from using irrational exponents? I think this is a fertile ground for significant future research.

In closing, I do not know for certain whether all of the $f(n)$ values listed here are truly the minimal values. In most cases, they were arrived at through computation and not through any mathematical theory. I look forward to hearing from readers who may be able to find even smaller values than the ones listed here. Finally, you may be interested in another contest conducted in 1989 called the "Very-large-number Contest," where participants were asked to construct an expression for a very large number using only the digits 1, 2, 3, and 4, and the symbols: "(," ")," decimal point, and the minus sign. Each digit could be used only once. The names of people who sent the 10 largest numbers were published in (Pickover, 1990, 1991).

Much of the participation and discussions for my *Terrible Twos Problem* occurred in the mathematics discussion group "sci.math" on the Usenet computer network, where this contest took place.

73.3 References

1. Pickover, C. (1990) Results of the very-large-number contest, *J. Recreational Math.* 22(4): 249-252. Also: Pickover, C. (1991) *Computers and the Imagination.* St. Martin's Press, New York
2. Guy, R. (1981) *Unsolved Problems in Number Theory.* Springer: New York.