Notations

(a) p_n denotes the n-th prime number.

(b) $S(n, k) = T(n, k) - T(n, k + 1)$, for all $n, k \in \mathbb{N}$ with $1 \le k \le row(n)$.

Observations

(c) $\sigma(2 \times p_n) = 1 + 2 + p_n + 2 \times p_n = 3 \times (p_n + 1)$.

(d) Every prime $p > 3$ satisfies $p = 6 \times k \pm 1$, for some $k \in \mathbb{N}$.

(e) $T(n, 1) = T(n - 1, 1) + 1$, for any $n \ge 1$.

(f) For any prime number $p > 3$:
$T(2 \times p, 1) = 2 \times p$ & $T(2 \times p - 1, 1) = 2 \times p - 1$
$T(2 \times p, 2) = T(2 \times p - 1, 2) = p - 1$

(g) Each leg in half of the n-th boundary path for the symmetric representation of $\sigma(n)$ has length $S(n, k)$, $1 \le k \le row(n)$

(h) If $S(n, 1) - S(n - 1, 1) = 1$, $S(n, 2) - S(n - 1, 2) = 0$, $S(n, 3) - S(n - 1, 3) = -1$ & $S(n, 4) > 0$, then the three pairs of legs $(S(n, i), S(n - 1, i))$, $i = 1, 2, 3$, together with the y-axis and the 4-th leg $S(n, 4)$ of the upper boundary define a closed, connected region of width one.

Claims

(1) For any prime number $p > 3$:

(a) $T(2 \times p, 3) = \begin{cases} \frac{2 \times p - 5}{3} & \text{if } p = 6k + 1 \\ \frac{2 \times p - 4}{3} & \text{if } p = 6k - 1 \end{cases} = T(2 \times p - 1, 3)$

(b) $T(2 \times p, 4) = \frac{p - 3}{2}$ & $T(2 \times p - 1, 4) = \frac{p - 5}{2}$

(c) $S(2 \times p, 4) = T(2 \times p, 4) - T(2 \times p, 5) > 0$

(2) For any prime number $p > 3$, $S(2 \times p, 1) - S(2 \times p - 1, 1) = 1$, $S(2 \times p, 2) - S(2 \times p - 1, 2) = 0$, $S(2 \times p, 3) - S(2 \times p - 1, 3) = -1$ & $S(2 \times p, 4) > 0$.

(3) For any prime number $p > 3$ such that $p \equiv -1 \bmod 6$, $\frac{S(2 \times p, 1)}{S(2 \times p, 2)} = 3$ and $\frac{S(2 \times p, 2)}{S(2 \times p, 3)} = 2$.

(4) The area of the region described in Observation (h) equals $\frac{3}{2} \times (p_{n+2} + 1) = \frac{1}{2} \times \sigma(2 \times p_{n+2})$.

(5) The sequence $2 \times p_{n+2}$, $n \ge 1$, is a sub-sequence of A239929.

Proof of (1.a), (1.b) & (1.c)

The corresponding proofs for $2 \times p - 1$ proceed similarly.

$$T(2 \times p, 3) = \left\lceil \frac{2 \times p + 1}{3} \right\rceil - 2 = \left\lceil \frac{2 \times (6 \times k \pm 1) + 1}{3} \right\rceil - 2 = 4 \times k + \left\lceil \frac{\pm 2 + 1}{3} \right\rceil - 2 = \begin{cases} 4k - 1 = \frac{2 \times p - 5}{3} & \text{if } p = 6k + 1 \\ 4k - 2 = \frac{2 \times p - 4}{3} & \text{if } p = 6k - 1 \end{cases}$$

$$T(2 \times p, 4) = \left\lceil \frac{2 \times p - 1}{4} \right\rceil - 2 = \left\lceil \frac{2 \times (6 \times k \pm 1) - 1}{4} \right\rceil - 2 = 3 \times k + \left\lceil \frac{\pm 2 - 1}{4} \right\rceil - 2 = \frac{p - 3}{2}$$

$$S(2 \times p, 4) = T(2 \times p, 4) - T(2 \times p, 5) = \left\lceil \frac{2 \times p + 3}{4} \right\rceil - 3 - \left\lceil \frac{2 \times p + 1}{5} \right\rceil + 3 = \left\lceil \frac{2 \times p + 3}{4} \right\rceil - \left\lceil \frac{2 \times p + 1}{5} \right\rceil \ge 1$$

Proof of (2)

From the Observations and Claim (1) we get:

$S(2\times p, 1) = T(2\times p, 1) - T(2\times p, 2) = 2\times p - (p - 1) = p + 1$

$S(2\times p - 1, 1) = T(2\times p - 1, 1) - T(2\times p - 1, 2) = (2\times p - 1) - (p - 1) = p$

$$S(2\times p, 2) = T(2\times p, 2) - T(2\times p, 3) = \left\{\begin{array}{ll} \frac{p+2}{3} & \text{if } p = 6k+1 \\ \frac{p+1}{3} & \text{if } p = 6k-1 \end{array}\right\} = \frac{2p+3}{6} \pm \frac{1}{6} = S(2\times p - 1, 2)$$

$$S(2\times p, 3) = T(2\times p, 3) - T(2\times p, 4) = \left\{\begin{array}{ll} \frac{p-1}{6} & \text{if } p = 6k+1 \\ \frac{p+1}{6} & \text{if } p = 6k-1 \end{array}\right\} = \frac{p}{6} \mp \frac{1}{6}$$

$$S(2\times p - 1, 3) = T(2\times p - 1, 3) - T(2\times p - 1, 4) = \left\{\begin{array}{ll} \frac{p+5}{6} & \text{if } p = 6k+1 \\ \frac{p+7}{6} & \text{if } p = 6k-1 \end{array}\right\} = \frac{p}{6} + 1 \mp \frac{1}{6}$$

Proof of (3)

Straightforward computations using the values from Claim (2) when $p \equiv -1 \bmod 6$ prove the assertion.

Proof of (4)

Claims (1) & (2) show that the area of the region described in Observation (h) can be computed just by the length of either its upper or its lower boundary. We get

$S(2\times p_{n+2}, 1) + S(2\times p_{n+2}, 2) + S(2\times p_{n+2}, 3) = T(2\times p_{n+2}, 1) - T(2\times p_{n+2}, 4)$

$= 2\times p_{n+2} - \frac{p_{n+2}-3}{2} = \frac{3\times p_{n+2}+3}{2} = \frac{1}{2}\times\sigma(2\times p_{n+2})$.

Proof of (5)

Claim (4) shows that the area of the symmetric representation of $\sigma(2\times p_{n+2})$ is $3\times(p_{n+2} + 1)$. Since the boundary path of the symmetric representation has length $4\times p_{n+2}$, there are exactly two regions.