Integers sequence A107991

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Abstract

A107991 is the sequence of the reciprocals of coefficients of a power series arising as the sum of repeated integrals of the hyperbolic cosine.

1 A classical result

Let $I = [a, b]$ be an interval such that $0 \in I$ and let $f\colon I \to \mathbb{R}$ be a continuous function.
Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of repeated integrals of f, i.e.

$$\begin{cases} f_0 = f \\ \forall x \in I,\ f_{n+1}(x) = \displaystyle\int_0^x f_n(t)\mathrm{d}t \end{cases}$$

A classical exercice (see e.g. *Le jardin d'Eiden – Une année de colles en Maths Spé*, Eiden Jean-Denis, *Calvage & Mounet*, 2012, p.510) is to prove that:

1. the functions series $\sum f_n$ is normally convergent on I;
2. its sum $F\colon x \mapsto \sum_{n=0}^{+\infty} f_n(x)$ can be written as :

$$\forall x \in I,\ F(x) = f(x) + \int_0^x f(t)\mathrm{e}^{x-t}\mathrm{d}t$$

2 Power Series of the sum of repeated integrals of the hyperboloic cosine

If $f = \cosh$, the sum F of its repeated integrals is $F\colon x \mapsto \dfrac{3\mathrm{e}^x + \mathrm{e}^{-x} + 2x\mathrm{e}^x}{4}$.
F can be written as a power series :

$$F(x) = 1 + \sum_{n=1}^{+\infty} \Big(\frac{3}{n!} + \frac{(-1)^n}{n!} + \frac{2}{(n-1)!}\Big)\frac{x^n}{4} = \sum_{n=0}^{+\infty} a_n x^n$$

where $a_0 = 1$ and $\forall n \in \mathbb{N}^*,\ a_n = \dfrac{2n + 3 + (-1)^n}{4\,n!}$.
We state that $\dfrac{1}{a_n} = \dfrac{4\,n!}{2n + 3 + (-1)^n}$ is equal to $\dfrac{n!}{\lfloor n/2 + 1\rfloor}$.

Indeed, if $n = 2m$ is even, then $\dfrac{1}{a_n} = \dfrac{1}{a_{2m}} = \dfrac{(2m)!}{m+1} = \dfrac{n!}{\lfloor n/2+1 \rfloor}$. And if $n = 2m+1$ is odd, then $\dfrac{1}{a_n} = \dfrac{1}{a_{2m+1}} = \dfrac{(2m+1)!}{m+1} = \dfrac{n!}{\lfloor n/2+1 \rfloor}$.

The sequence $\left(\frac{1}{a_n}\right)_{n\in\mathbb{N}} = (1, 1, 1, 3, 8, 40, 180, 1260, 8064, \ldots)$ corresponds to the A107991 OEIS sequence.