A049310: Chebyshev S-polynomial coefficient table
10 applications of these polynomials.
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A049310 tabulates the coefficient array of the integer monic Chebyshev S-polynomials
S(n,x):=U(n,x/2), with the usual Chebyshev U-polynomials of the second kind.
These polynomials appear in many applications, and I will list 10 points.
(Point (1) was used in the sequence of the day:
http://oeis.org/wiki/Template:Sequence_of_the_Day_for_November_15)

(1) In linear atomic chains with N uniformly harmonic interacting atoms of the same mass. The eigenmodes have scaled frequency squares x given by the zeros of S(N,2(1-x)).
The recurrence for the oscillations with frequency omega and displacement q_n from the equilibrium position at site no. n, q_n(t)= q_n*exp(i omega t) (i is the complex unit) , is:

q_{n+1} - 2 (1-x) q_n + q_{ n-1} = 0.

Here x:= omega^2/(2 (omega_0)^2), the normalized frequency squared, with omega_0:=k/m, where k is the uniform spring constant and m the atom's mass. This leads to the so called 2x2 transfer matrix R(x):=[[2(1-x),-1],[1,0]], with [q_{n+1},q_{n}]^T = R(x) [q_{n},q_{n-1}]^T (T for transposed).
Iteration yields

[q_{n+1},q_{n}]^T = M_n(x) [q_{1},q_{0}]^T,

with M_n(x)=R(x)^n, and the two arbitrary inputs q_{1} and q_{0}. It follows that

M_n(x) = [[S(n,2*(1-x)),S(n-1,2*(1-x))],[-S(n-1,2*(1-x)),-S(n-2,2(1-x))]]

due to the recurrence for the S- polynomials:

S(n,x) = x *S(n-1,x) - S(n-2,x), S(-1,x)=0, S(0,x)=1, n>=1.

Thus one obtains the general solution for the displacements

q_{n+1} (x)= S(n,2(1-x))* q_1 - S(n-1,2(1-x))* q_0.

For finite N-chains with fixed boundary conditions q_0= 0 = q_{N+1} one therefore has to solve S(N,2(1-x))=0, and thus obtains the N normalized eigenfrequency squares for the N-chain:

x^{(N)}_{k} = 2 (sin(Pi*k/(2*(N+1))))^2, k=1,...,N.

A side remark: because Det R(x)=1, also Det M_n(x) =1, identically, therefore on has the so called Cassini-Simson identity

(S(n-1,y))^2 - S(n,y)*S(n-2,y) = 1, for each n>=0.

(2) In graph theory S(N,x) is the characteristic polynomial of the P_N graph (n vertices, n-1 edges)

with tridiagonal adjacency matrix [[0100...],[1010...],[01010..],...[...010]]. See the Michael Somos comment. For instance, the ordinary generating function for n(P_n,L), the total number of closed paths (walks) of length L on the graph P_N denoted by W(P_N,x) := sum(n(P_N,L) x^L, L=0..infinity) turns out to be y diff(S(N,y),y)/S(N,y) with y=1/x. Note that this has to produce vanishing n(P_n,L) for odd L . This can be rewritten as

W(P_N,x) = ((N+1)*coth((N+1)*ln(2*x/(1-sqrt(1-(2*x)^2)))) - 1/sqrt(1-(2*x)^2))/sqrt(1-(2*x)^2).

For example, if N=4, one obtains the sequence [4, 0, 6, 0, 14, 0, 36, 0, 94, 0,...] (2*A198634 without the zeros). This computation uses the moments (powers) of the zeros of the characteristic polynomial S(N,x) for the adjacency matrix of the P_N graph. See the array A198632 and comments there. S(N,x) is also the matching polynomial for the P_N graph. See C. D. Godsil, Algebraic Combinatorics, p. 2, Exercise 1, p. 14, and p. 144.

(3) In matrix theory one finds for the power of every 2x2 matrix with Det(M) = y, not 0, and trace(M) = x, due to the Cayley-Hamilton theorem M^2 - x M + y I = 0 (I is the 2x2 identity matrix), for the general power (note that S(n,-2)= -1)

M^n = (sqrt(y))^(n-1)*(S(n-1,x/sqrt(y))*M - sqrt(y)*S(n-2,x/sqrt(y))*I), n=0,1,....

This application lends itself to a generalization to more variable Chebyshev polynomials when one takes nxn matrices.
As a corollary, consider the 2x2 matrix representation of a complex number z=x+y i by Z:=[[x,y],[-y,x]] with Det(Z) = x^2+y^2 =: rho^2 (not 0) and Trace(Z) = 2x (see, e.g., R. Remmert, Komplexe Zahlen, in H.-D. Ebbinghaus et al., Zahlen, 1992, Springer, Kap. 3, p. 53). Then the usual de Moivre formula is replaced by

Z^n = rho^(n-1) *(S(n-1,2*cos(phi))*Z - rho*S(n-2,2*cos(phi))*I),

with cos(phi):=x/rho.

(4) The relation to Fibonacci numbers F(n)=A000045(n) and Lucas numbers L(n)=A000032(n) is, due to their recurrences with inputs (i is again the complex unit):

F(n) = (-i)^(n-1)*S(n-1,i) and L(n) = 2*(-i)^n* T(n,i/2),

with the so called trace polynomials (see the trace of the transfer matrix M_n(x) from part (1) divided by 2)

T(n,x):=(S(n,x) - S(n-2,x))/2

(they turn out to be the Chebyshev polynomials of the first kind, with coefficients given in A053120). One has, in particular, F(2(n+1)) = S(n,3), F(2n-1) = S(n-1,3)-S(n-2,3), F(3n) = 2 (-1)^{n-1} S(n-1,4 i) and L(3n)=2 (-1)^n T(n,2i).

(5) {S(n,x)}_0^infinity is a monic orthogonal polynomial system (mOPS) in the variable x from the interval [-2,+2], with weight function w(x) = sqrt(1-(x/2)^2) (for the U-polynomials, orthogonal on the interval [-1,+1] the weight function is sqrt(1-x^2), called sometimes the Wigner semicircle (plot it to see why)). This is a classical orthogonal polynomial system, implying, that besides the three term

recurrence it also satisfies a differential equation of the hypergeometric type. It belongs to the class of Jacobi polynomials {P^{(alpha,beta)}(n,x)} if one puts alpha=1/2 = beta and replaces x by x/2:

S(n,x) =((n+1)!/risefac(3/2,n)) P^{(1/2,1/2)}(x/2) = C^{(1)} (x/2)

with the ultra-spherical or Gegenbauer polynomials C^(lambda)}(x) (risefac(x,n) := x(x+1)...(x+(n-1))). This hypergeometric differential equation is:

(4-x^2)*diff(S(n,x),x$2) - 3*x*diff(S(n,x),x) + n*(n+2)*S(n,x) = 0, n>=0.

This orthogonal polynomial connection is intimately tied to continued fractions. The n-th approximation to the Jacobi continued fraction for the S-polynomials is

J(n,x) = 1/(x-1/(x-1/(x- ...) n brackets) = S(n-1,x)/S(n,x), n>=1.

The S-polynomials are called the denominator polynomials of the continued fraction approximation, and here the numerator polynomials turn out to be also S-polynomials because their recurrence coefficients are n-independent.

(6) In q- (or basic) analysis one uses the q-numbers

[n]_q = (q^n - (1/q)^n)/(q - 1/q) = S(n-1,q+1/q),

which reduce to n for q -> 1 (use the finite geometric series or l'Hospital's rule). This is just the well known Binet-de Moivre form of the S-polynomials, obtained from their ordinary generating function by expansion.

(7) In the study of Diophantine equations the so called Pell equation x^2 - d y^2 =1, with square-free d, has general solution

x_k = T(k+1,x_1) = (S(k+1,2*x_1) - S(k-1,2*x_1))/2 and y_k = y_1*S(k,2*x_1),

with the solution (x_1,y_1) with the smallest x_1 not +/-1. E.g., d=5 has (x_1,y_1) = (9,4). For (x_1(d),y_1(d)) see the sequences (A033313,A033317).

(8) In the theory of algebraic numbers a factorization of S(n-1,x) over the rationals appears in the recursive definition of the minimal polynomials Psi(n,x) for the algebraic number cos(2 pi/n), n>=1. See the Watkins and Zeitlin reference given in A181875, where one finds also a link listing these minimal polynomials. The formula is

S(n-1,x) = 2^(n-1)*product(Psi(d,x/2), 2<d dividing 2n), n>=1.

Similarly,

S(2n,sqrt(2-x)) = 2^n*product(Psi(d,x/2), 1<d dividing 2n+1), n>=0.

(9) For applications in approximation theory for functions see the Th. Rivlin book (reference given under this triangle A049310), where the monic T_n-polynomials feature because of their minimal property to have among the monic real polynomials of degree n on the interval [-1,+1] the smallest

maximal absolute value (deviation from 0).

(10) Finally, the most important property of the S-polynomials which is responsible for many of their properties is that their coefficients lead to an (ordinary) convolution triangle, meaning that, from the second column on, consecutive columns are obtained via convolution. This is a result of the structure of the column o.g.f.s

G(m,x) = G(x)*(x*Fhat(x))^k, k>=0,

with G and Fhat some formal power series starting with 1. Such convolution matrices (with zeros filling all entries above the diagonal of the triangle) are called Riordan matrices, denoted by (G(z),F(z)), with F(z)=z*Fhat(z). The simplest nontrivial example (the trivial one is the unit matrix with the monomials x^n as row polynomials) for such a convolution triangle is, of course, Pascal's triangle A007318. The ordinary generating function (o.g.f.) of the row polynomials R(n,x) is then G(z,x) := G(z)/(1 - x F(z)). Under matrix multiplication this convolution property is preserved and one speaks of the Riordan group. See the L. W. Shapiro, et al. reference given under A007318 . In our case G(z)=Fhat(z)=1/(1+z^2), and this belongs to a subgroup, called the (E. T.) Bell group (check that the o.g.f. is indeed the one for the S-polynomials).
Therefore the o.g.f.s for the column no. m, m>=0, sequences are:

G(m,x) = x^m/(1+z)^(m+1).

The o.g.f. for the row sums is R(n,x) = 1/(1-x +x^2), which generates the periodic(1,1,0,-1,-1,0) sequence found as A010892, and the one for the alternating row sums is aR(x)= 1/(1+x +x^2), which generates periodic(1,-1,0) found as A049347. It also leads to the o.g.f. for the so-called A- and Z-sequences (see A06322, with a W. Lang link with the definition of A- and Z-sequences for Riordan arrays and references, especially those for D. G. Rodgers and D. Merlini et al.) A(x)=1-x^2*c(x^2), with c(x) the o.g.f. for the Catalan numbers A000108, which generates [1,0,-1,0,-1,0,-2,0,...], and the one for the Z-sequence Z(x) = -x*c(x^2), which generates -[0,1,0,1,0,2,0,0, 5,...] found under -A1261290. These A- and Z-sequences yield a recurrence relations for the Riordan (or Bell) matrix entries (see the formula section of the triangle) which is different from the usual one given also there.

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