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it can be shown for large n that Y_{n-1} is approximately equal to $(2\pi npq)^{-1/2}$, or $1/\sqrt{2\pi}\sigma_n$. Thus from (6) we obtain the approximation

$$MD_n \sim \sqrt{2npq/\pi} = \sqrt{2/\pi}\,\sigma_n = 0.79788\sigma_n. \tag{11}$$

More exact computation, using the remainder terms in Stirling's formula, yields the better approximation

$$\frac{\pi}{2}(MD_n)^2 = npq + (np - [np])(nq - [nq]) - (1 - pq)/6 + E_n/24n, \tag{12}$$

where the error coefficient E_n becomes numerically less than or equal to unity as n becomes infinite, for all choices of np between 1 and $n-1$; and $[np]$ and $[nq]$ denote the greatest integers not exceeding np and nq respectively.

A SET OF EIGHT NUMBERS

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1. Introduction. In this paper the operation of adding the squared digits of any natural number A a finite number of times is proved to transform A either to unity or to one of a set of eight natural numbers closed under the operation.

2. Definitions. We use the expression *natural number* to denote a member of set 1, 2, 3, · · · of positive integers. Zero has not been adjoined to this set d is not to be included in the definition.

The operator G is defined by the equation

$$G(A) = \sum_{i=1}^{R} X_i^2, \tag{1}$$

where A is a natural number of R digits given by

$$A = \sum_{i=1}^{R} X_i 10^{i-1}. \tag{2}$$

Since A has R digits, $X_R \neq 0$.

We note that $G(0)=0$, and $G(1)=1$.

Using the customary notation, we write $G^n(A)$, where $n>1$, for n successive applications of the operator G to A.

G is not a linear operator since, in general, $G(A_1+A_2) \neq G(A_1)+G(A_2)$.

The set of numbers

$$\begin{aligned} a_1 &= 4, & a_5 &= 89, \\ a_2 &= 16, & a_6 &= 145, \\ a_3 &= 37, & a_7 &= 42, \\ a_4 &= 58, & a_8 &= 20, \end{aligned} \tag{3}$$

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is closed under the operation defined by (1). We call (3) *Set* K, and use the symbol a' to denote any non-specified element of the set. The equation

$$G^8(a') = a' \tag{4}$$

is easily verified.

Numbers of the form 10^n, $13 \cdot 10^n$, $10^{n+1}+3$, where n is a positive integer or zero, and others not specified here, satisfy the equation

$$G^r(A) = 1 \tag{5}$$

for some integer $r>0$. Any natural number satisfying (5) will be denoted by the symbol b'.

3. Preliminary Lemmas. In what follows, the symbols A and B always represent finite natural numbers in the denary system of notation.

LEMMA 1. *Any natural number A of R digits, where $R \geqq 4$, satisfies the inequality*

$$G(A) < A. \tag{6}$$

It is evident that $G(A) \leqq 81R$, and that $A \geqq 10^{R-1}$. The inequality

$$81R < 10^{R-1} \tag{7}$$

becomes, upon taking the common logarithm of each member and transposing,

$$\log_{10} R < R - 2.9085, \tag{8}$$

an inequality valid for $R \geqq 4$.

LEMMA 2. *For any natural number A there exists a positive integer n such that*

$$G^n(A) \leqq 162. \tag{9}$$

For $R \geqq 4$, Lemma 1 establishes the inequality (6). As a direct consequence of (6), the operator G applied to A a finite number of times must result in a natural number of less than four digits, since for $R=4$, $G(A) \leqq 324$.

For $R<4$, the following inequalities are readily established.

$$G(A) \leqq 243, \tag{10}$$

$$G^2(A) \leqq G(199) = 163, \tag{11}$$

$$G^3(A) \leqq G(99) = 162. \tag{12}$$

Since $G(A)$, where A is a three digit number, cannot exceed $3 \cdot 81 = 243$, (10) is obviously valid. Also, since $G(199) \geqq G(B)$ for any $B \leqq 243$, (11) holds. Finally, since $G(99) \geqq G(P)$ for any $P \leqq 163$, (12) is proved.

The inequalities (10), (11), and (12) complete the proof of Lemma 2.

4. Convergence of $G^n(A)$. The following theorem is the main result of this paper.

THEOREM 1. *For every natural number A there exists either a positive integer n such that* (5) *holds for all* $r \geqq n$, *or a positive integer m such that*

$$G^r(A) = a' \tag{13}$$

for all $r \geqq m$, *where* a' *is some element of Set K.*

From Lemma 2 it is evident we need prove the theorem only for $A \leqq 162$. The writer was unable to find a simple indirect proof sufficiently superior to the following direct one of selective verification to justify its inclusion here.

We consider two cases.

Case 1. $100 \leqq A \leqq 162$.

For A thus restricted, it is apparent that $G(A) \leqq G(159) = 107$. Direct application of the operator G to A over the range 100 to 107 gives

$$\begin{aligned} G(100) &= 1, & G^6(104) &= a' = 89, \\ G^2(101) &= a' = 4, & G^3(105) &= a' = 16, \\ G^5(102) &= a' = 89, & G(106) &= a' = 37, \\ G^2(103) &= 1, & G^5(107) &= a' = 89, \end{aligned} \tag{14}$$

thus completing the proof of the theorem for Case 1.

Case 2. $0 < A < 100$.

For $A = 10X + Y$, where $0 \leqq X \leqq 9$, and $0 \leqq Y \leqq 9$, the following identity is valid.

$$G(10X + Y) \equiv G(10Y + X). \tag{15}$$

Further, if $G^n(A) = a'$, and $G^m(B) = A$, it follows that there exists a number $h = n + m$ such that $G^h(B) = a'$.

By means of these considerations, it is possible to verify Theorem 1 numerically for all $A < 100$ by actual computation of $G^n(A)$ for 30 values of $A < 100$, thus completing the proof of the theorem.

The writer is aware of the inelegance of such a proof, and would like very much to see a simple indirect one. However, proving the non-existence of another set like (3), which seems a necessary step, is quite difficult because of the non-linear character of G.

COROLLARY. *For every natural number A there exists either a positive integer n such that* $G^n(A) = 1$, *or a positive integer m such that* $G^m(A) = 4$.

The corollary follows directly from Theorem 1 and the nature of Set K. Since every natural number is transformed either into unity or into an element of Set K by the operator G, we need only note that for every $a' \neq 4$, there exists a positive integer $r \leqq 7$ such that $G^r(a') = 4$.

THEOREM 2. *The number of digits N in G(A), where A has R digits, satisfies the inequality*

$$N \leqq 2.9 + \log_{10} R. \tag{16}$$

This theorem is a simple consequence of the inequality $G(A) \leqq 81R$. We have

$$G(A) \leqq 10^{1.9} + \log_{10} R \tag{17}$$

a number of N digits, where $N \leqq 2.9 + \text{Log}_{10} R$.

THEOREM 3. *The only solutions in natural numbers of*

$$G^n(A) = A,$$

where $n \geqq 1$, *are*

$$A = 1, \qquad n = J, \tag{19}$$

$$A = a', \qquad n = 8, \tag{20}$$

where J *is any natural number.*

If we assume the existence of a natural number $A > 1$ and different from a' such that $G^n(A) = A$ for some $n \geqq 1$, it follows that A would not be transformed into either unity or an element of Set K by a finite number of applications of the operator G to A. But this is a direct contradiction of Theorem 1, and hence the assumption is false.

5. Concluding Remarks. A problem suggested by the one just discussed is that of repeatedly summing the *cubed* digits of a natural number. A complication occurs, however, since there is more than one number A such that $H(A) = A$, where H is the operator analogous to (1) given by

$$H(A) = \sum_{i=1}^{R} X_i^3. \tag{21}$$

For example, $H(153) = 153$, $H(407) = 407$, and $H(371) = 371$. This destroys the factor of uniqueness, since $H(A)$ may be unity as when $A = 100$; or A may be transformed into a number A' like 153.

It is interesting to note that since for any number A transformed into some element of Set K by a finite number of applications of G we can construct a number $B = 10^A$ such that $G(B) = 1$, there are at least "as many" numbers satisfying (5) as (13). This intuitionally unsatisfying conclusion results from the comparison of two infinite sets.

Leibniz discovers the obvious. I have made some observations on prime numbers which, in my opinion, are of consequence for the perfection of the science of numbers . . . . If the sequence [of primes] were well known, it would enable us to uncover the mystery of numbers in general; but up till now it has seemed so bizarre that nobody has succeeded in finding any affirmative characteristic or property . . . . I believe I have found the right road for penetrating their [primes'] nature: but not having had the leisure to pursue it, I shall give you here a positive property, which seems to me curious and useful.—Leibniz, in a letter to the editor of the *Journal des Savans*, 1678. The discovery: a prime is necessarily of one or other of the forms $6n+1$, $6n+5$.—*Contributed.*