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CONSTRUCTION AND ENUMERATION OF REGULAR MAPS ON THE TORUS

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Abstract. A construction is given for all the regular maps of type $\{3,6\}$ on the torus, with v vertices, v being any integer > 0. We also find bounds for the number of those maps, in particular for the case in which the maps contain "normal" Hamiltonian circuits. Using duality, the results may be applied for the maps of type $\{6,3\}$ too.

1. Introduction

The terms used here are commonly used in graph theory; however, all of them are defined in [2]. A map on the torus is a cellular decomposition of the torus imposed by some graph, the *graph of the map*. The map is *regular of type* $\{p,q\}$ if all the cells (the *faces* of the map) have the same number $p \geq 3$ of edges and all the vertices have the same valence $q \geq 3$, i.e., for every vertex x in the map, there are exactly q edges incident to x (a loop incident to x is counted twice in this connection).

Let T be a regular map of type $\{p,q\}$ on the torus, and let v, e, f be the numbers of its vertices, edges and faces, respectively. From the identities $q \cdot v = 2e = p \cdot f$ and from Euler's equation $v - e + f = 0$, it can be easily deduced that $\{p,q\}$ may have the values $\{3,6\}$, $\{4,4\}$ and $\{6,3\}$ only.

An edge with the vertices a, b we denote by (ab), an n-gonal face with the edges $(a_1a_2), (a_2a_3), ..., (a_na_1)$ we denote by $(a_1a_2, ..., a_n)$, and a path through the vertices $a_1, a_2, ..., a_m$ in this order we denote by $(a_1, a_2, ..., a_m)$. A map T_1 is *isomorphic* to a map T_2 ($T_1 \approx T_2$) if there is a one-to-one correspondence φ between the vertices of T_1 and the vertices of T_2, such that (ab) is an edge in T_1 if and only if $\varphi(ab) =$

$(\varphi(a)\,\varphi(b))$ is an edge in T_2, and $(a_1 a_2, \ldots, a_n)$ is a face in T_1 if and only if $\varphi(a_1, \ldots, a_n) = (\varphi(a_1)\varphi(a_2), \ldots, \varphi(a_n))$ is a face in T_2.

In this paper, we investigate for every natural number v the constructions and the number of all non-isomorphic regular maps of types $\{3,6\}$ and $\{6,3\}$ on the torus, with v vertices. The map T^* dual to a map T of type $\{p,q\}$ is obviously of type $\{q,p\}$, and for maps T_1, T_2 of the same type, $T_1 \approx T_2$ holds if and only if $T_1^* \approx T_2^*$. For a map of type $\{3,6\}$, the equations $e = 3v$, $f = 2v$ hold, and the number of all different (i.e., non-isomorphic) maps of type $\{3,6\}$ with v vertices equals the number of all different maps of type $\{6,3\}$ with v faces (all of them hexagons), being at the same time all different maps of type $\{6,3\}$ with $2v$ vertices. (It follows immediately that there is no map of type $\{6,3\}$ with an odd number of vertices.) Hence it is sufficient to carry out our investigations for the maps of type $\{3,6\}$ only. Therefore, all of our theorems deal with maps of type $\{3,6\}$, leaving to the reader the formulation of the dual theorems.

In Section 2, we describe a construction which gives, for any integer v, all the regular maps of type $\{3,6\}$ with v vertices, which have a normal circuit (defined in Section 2) through all the vertices (a normal Hamiltonian circuit). We also find when such two constructions yield the same map.

The same investigation for the general case, in which the map not necessarily includes a normal Hamiltonian circuit, was carried out by the author in a similar method, but the calculations in this case are much longer and much more complicated. Therefore we give in Section 3 only the results of these investigations. The author will supply the detailed proofs to every interested reader.

In Section 4, we discuss the number $\chi(v)$ of different regular maps of type $\{3,6\}$ with v vertices, and we find bounds for the number $\lambda(v)$ of those maps which have a normal Hamiltonian circuit.

A (regular) map is called also a (regular) *polygonization* if there are no loops and double edges in the graph of the map, and the intersection of any two faces in the map is an edge, a vertex, or empty. Hence the set of all faces, edges and vertices of a polygonization forms a topological complex. The map dual to a polygonization is also a polygonization. A polygonization all of whose faces are triangles is called a *triangulation.*

A good reason for distinguishing between maps and polygonizations is, that a map which is not a polygonization is clearly not rectilinearly embeddable in $\mathbf{R}^3$, while many regular triangulations of type {3, 6} on the torus yield such an embedding (see [1, Theorem 7; 2, Theorem 3]), and it is not known whether there exists any triangulation which does not yield such an embedding [7].

What is the minimal number of vertices in a polygonization of the torus? Clearly, in such a map all the faces are triangles. Euler's formula yields for such a map: $e = 3v$, $f = 2v$. We also have $e \leq \binom{v}{2}$, i.e. $3v \leq \binom{v}{2}$, and this yields

$$v \geq 7, e \geq 21, f \geq 14,$$

and equality in one of these yields an equality in the others too.

A polygonization K of the torus with 7 vertices does in fact exist, and it is regular of type {3, 6}. Its graph is the complete graph on seven vertices. The map K^* dual to K is known in the literature as a map which needs seven colors for coloring it. (See, e.g., [4, Section 4.6].) As first discovered by Möbius [9, pp. 552, 553] and rediscovered by Császár [6], the linear embedding of this map K yields a polyhedron which, like the simplex, has no diagonals.

In Sections 2 and 3, we discuss both maps and triangulations, and the notes which refer to triangulations we enclose by brackets [ ].

Our use of the term "regular map" agrees with Erréra [8] and Threlfall [10], but not with Brahana [3] and Coxeter-Moser [5, Chapter 8]. The three last authors are interested in regular maps from the standpoint of group theory, and their methods do not seem useful for our purpose.

2. Maps with a normal Hamiltonian circuit[1]

The notion of a *normal path* in a map of type {6, 3} is essential for the study of the combinatorial structure of those maps. Let T be a map of type {6, 3} on the torus, L a path in T, and x an inner vertex in L. L is *normal at* x if of the four edges incident to x and not on L, two are on one side of L, and the other two edges are on the other side of L. The

[1] This section is included in the author's Ph. D. dissertation, written under the supervision of Professor H. Furstenberg, and submitted to the Hebrew University in June 1969.

path L is *normal* if it is normal at every inner vertex. (A more rigorous definition is given in [2].) It is by no means obvious that producing a normal path in a "normal" manner as far as possible we get a normal circuit. However, this is the case [2, Theorem 1], and hence it follows immediately that considering the three different normal circuits through an arbitrary vertex of T, any other normal circuit is "parallel" to one of those three circuits and has the same length.

Here we investigate, for any given v all the regular maps and triangulations of type $\{3,6\}$ on the torus, with v vertices and with a normal Hamiltonian circuit.

Consider a regular map [triangulation] of type $\{3,6\}$ on the torus with v vertices and with a normal circuit L through all its vertices. Denote the vertices $a_1, ..., a_v$ according to their order on L. Let m be minimal such that a_1 is joined to a_m by an edge not on L. [For triangulation we have: $m \neq 1, 2, v$.] Together with $(a_1 a_m)$ there is another edge A emerging from a_1 on the same side of L. It can be easily seen that each triangle in the map has exactly one edge on L, hence it follows $A = (a_1 a_{m+1})$. Repeated application of this reasoning gives that, for every $1 \leq i \leq v$, $(a_i a_{i+m-1})$ and $(a_i a_{i+m})$ (all the indices are mod v and a_0 is a_v) are edges of the map and together with the edges on L they form all the edges of the map. Therefore, we can represent our map as shown in Fig. 1, and we denote this representation by $T_m^{v,1}$. If there is no doubt about v, we write T_m. (In Fig. 1, the identification of the vertical sides of the rectangle is in the natural manner, while the identification of the lower and the upper sides is according to the notation of the vertices, and needs some shifting.)

The vertices in T_m adjacent to a_i $(1 \leq i \leq v)$ are

$$a_{v-m+i+1},\ a_{v-m+i},\ a_{i+m},\ a_{i+m-1},\ a_{i+1},\ a_{i-1}\ .$$

[In a triangulation those vertices are different from each other and dif-

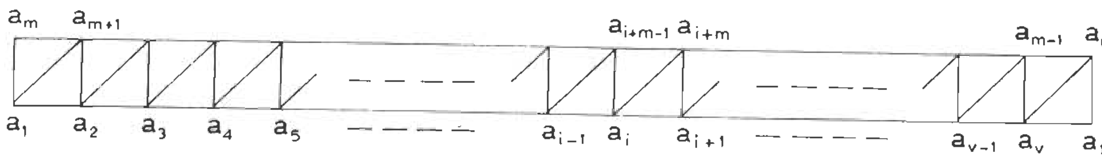

Fig. 1.

ferent than a_i. Hence we have $m \neq 1, 2, v$, which we know already, and we have also

$$2m \neq v+1,\; m \neq v-1,\; 2m \neq v+2,\; 2m \neq v.$$

Therefore, if $v \leq 6$, there is no such m at all, which was foreseen from a note in Section 1.]

The trinagles in T_m are

$$\{(a_i a_{i+1} a_{i+v+1-m}),\, (a_i a_{i+1} a_{i+m}) \mid 1 \leq i \leq v\}$$

and those are also the triangles in T_{v-m+1}. Hence $T_{v-m+1} \approx T_m$. Therefore it is sufficient to consider the case $1 \leq m \leq \frac{1}{2}(v+1)$. [For triangulation: $1 \leq m < \frac{1}{2}v$. In particular, for $v = 7$ we have only $m = 3$ (even without demanding the existence of a normal circuit through all the vertices. In this case, as for every prime v, its existence is assured (Consequence 3.1), thus proving the uniqueness of the triangulation of Császár].

As usual, (v, m) denotes the greatest common divisor of v and m.

Theorem 2.1. (i) *For every regular map [triangulation] of type* $\{3,6\}$ *on the torus with* v *vertices and with a normal circuit through all the vertices, there is an integer* m*,* $1 \leq m \leq \frac{1}{2}(v+1)$ $[3 \leq m < \frac{1}{2}v]$*, such that the map [triangulation] is isomorphic to* T_m.

(ii) *For* $1 \leq m_1 \neq m_2 \leq \frac{1}{2}(v+1)$ $[3 \leq m_1 \neq m_2 < \frac{1}{2}v]$, $T_{m_1} \approx T_{m_2}$ *holds if and only if one of the following congruences* (1)–(4) *is satisfied.*

(1) $m_1 \cdot m_2 \equiv 1 \pmod{v}$,

(2) $m_1 \cdot (m_2 - 1) \equiv -1 \pmod{v}$,

(3) $(m_1 - 1)m_2 \equiv -1 \pmod{v}$,

(4) $(m_1 - 1)(m_2 - 1) \equiv 1 \pmod{v}$.

Proof. Part (i) of the theorem follows from the previous notes. We prove the second part. First we prove the necessity of the condition.

Assume there is an isomorphism ϕ: $T_{m_1} \to T_{m_2}$, and for each $1 \leq i \leq v$ denote $\phi(a_i) = b_i$. For every k $(1 \leq k \leq v)$, the function $\psi(a_i) = a_{k+i}$ $(1 \leq i \leq v)$ naturally induces an automorphism of T_{m_2}, hence

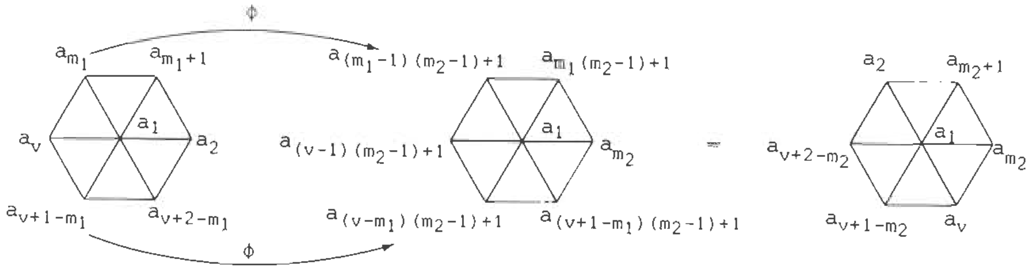

Diagram 1.

we can assume, without loss of generality, that $b_1 = a_1$. There are six possibilities for b_2:

(a) $b_2 = a_2$, (b) $b_2 = a_{m_2}$, (c) $b_2 = a_{m_2+1}$,
(d) $b_2 = a_v$, (e) $b_2 = a_{v+2-m_2}$, (f) $b_2 = a_{v+1-m_2}$,

since b_2 is adjacent to a_1 in T_{m_2} and the above six vertices are the only vertices adjacent to a_1 in T_{m_2}. The function $\eta(a_i) = a_{v+2-i}$ $(1 \le i \le v)$ naturally induces an automorphism of T_{m_2}. Hence it is sufficient to examine the cases (a), (b), (c).

Case (a): $b_2 = a_2$. Then clearly $b_i = a_i$ for every $1 \le i \le v$. Therefore, the triangle $(a_1 a_2 a_{m_1+1})$ in T_{m_1} is transformed into $(a_1 a_2 a_{m_1+1})$ in T_{m_2}. But, in T_{m_2} the only two triangles with the edge $(a_1 a_2)$ are $(a_1 a_2 a_{m_2+1})$ and $(a_1 a_2 a_{v+2-m_2})$. Hence $m_1 + 1 = m_2 + 1$, contradicting the assumption $m_1 \neq m_2$, or $m_1 + 1 = v + 2 - m_2$, i.e., $m_1 + m_2 = v + 1$ contradicting the assumption $1 \le m_1 \neq m_2 \le \frac{1}{2}(v+1)$. Therefore case (a) is impossible.

Case (b): $b_2 = a_{m_2}$. Then clearly $b_i = a_{(m_2-1)(i-1)+1}$ for every $1 \le i \le v$. Assume the Diagram 1.

In the two hexagons on the left there is a correspondence between the vertices, given by the diagram. In the two hexagons on the right, there is not necessarily such a correspondence. It may happen for $a_{m_1(m_2-1)+1}$ to be equal a_{m_2+1} or a_v.

Therefore, if $a_{m_2+1} = a_{m_1(m_2-1)+1}$, then $m_2 + 1 \equiv m_1(m_2-1)+1 \pmod v$, i.e., $(m_1-1)(m_2-1) \equiv 1 \pmod v$ and we have the congruence (4), and if $a_v = a_{m_1(m_2-1)+1}$, then $v \equiv (m_2-1)m_1 + 1 \pmod v$, i.e., $m_1(m_2-1) \equiv -1 \pmod v$ and we have the congruence (2).

Case (c): $b_2 = a_{m_2+1}$. The examination is as in case (b). Here, $b_i =$

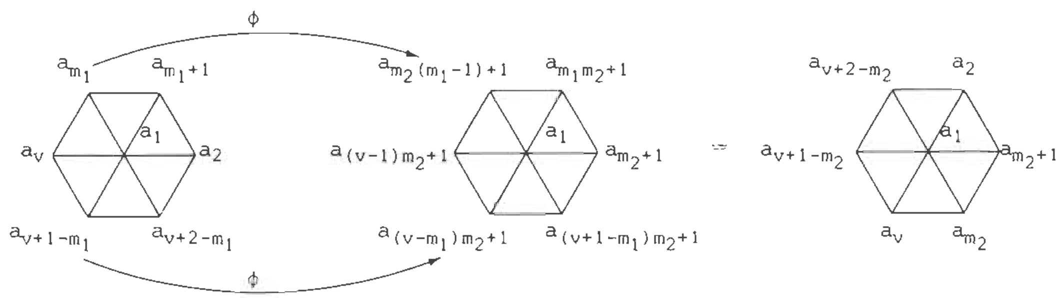

Diagram 2.

$a_{m_2(i-1)+1}$ for every $1 \leq i \leq v$. Hence the situation is as in Diagram 2. Therefore, if $a_2 = a_{m_1 m_2+1}$, then $m_1 m_2 + 1 \equiv 2 \pmod v$, i.e., $m_1 m_2 \equiv 1 \pmod v$ and this is the congruence (1), and if $a_2 = a_{(v+1-m_1)m_2+1}$, then $(v+1-m_1)m_2 + 1 \equiv 2 \pmod v$, i.e., $m_2(m_1-1) \equiv -1 \pmod v$ and this is the congruence (3).

Sufficiency. Assume that $1 \leq m_1 \neq m_2 \leq \frac{1}{2}(v+1)$ $[3 < m_1 \neq m_2 < \frac{1}{2}v]$ and that one of the congruences (1)–(4) is satisfied. We show that this implies $T_{m_1} \approx T_{m_2}$. The triangles of T_{m_1} are

$$\{(a_i a_{i+1} a_{i+m_1}), (a_i a_{i+1} a_{v+i+1-m_1}) \mid 1 \leq i \leq v\}.$$

In the case that (1) is satisfied, let $\phi(a_i) = a_{m_2(i-1)+1}$, and we show that ϕ naturally induces the desired isomorphism from T_{m_1} to T_{m_2}. The triangles of T_{m_1} are mapped by ϕ to

$$(*) \qquad \{(a_{m_2(i-1)+1} a_{m_2 i+1} a_{m_2(i+m_1-1)+1}), (a_{m_2(i-1)+1} a_{m_2 i+1} a_{m_2(v+i-m_1)+1}) \mid 1 \leq i \leq v\}.$$

Let $j = 1 + (i-1)m_2$. Since $(v, m_2) = 1$ holds, then, when i ranges over the numbers 1, 2, ..., v, also $j \pmod v$ ranges over those numbers. Then

$$\begin{aligned} m_2(i-1)+1 &= j, & m_2(i+m_1-1)+1 &= j+m_1 m_2 \equiv j+1 \pmod v, \\ m_2 i+1 &= j+m_2, & m_2(v+i-m_1)+1 &= j+m_2(v-m_1)+m_2 \\ & & &\equiv j+m_2-1 \pmod v. \end{aligned}$$

Hence the set (*) is identical with the set

$$\{(a_j a_{j+m_2} a_{j+1}), (a_j a_{j+m_2} a_{j+m_2-1}) \mid 1 \leq j \leq v\}$$

and denoting $t \equiv j+m_2-1 \pmod{v}$, this is identical with the set

$$\{(a_j a_{j+1} a_{j+m_2}) \mid 1 \le j \le v\} \cup \{(a_t a_{t+1} a_{t+1-m_2}) \mid 1 \le t \le v\} ,$$

which is exactly the set of all triangles of T_{m_2}. Hence $T_{m_1} \approx T_{m_2}$. In the case that congruence (3) is satisfied, let $\phi(a_1) = a_{m_2(i-1)+1}$ and let j be defined as before. Then

$$\begin{aligned} 1+im_2 &= j+m_2, \\ 1+(i+m_1-1)m_2 &= j+m_1m_2 \equiv j+m_2-1 \pmod{v} , \\ 1+(v+i-m_1)m_2 &= j+m_2+(v-m_1)m_2 \equiv j+1 \pmod{v} , \end{aligned}$$

and, as before, the set of all triangles of T_{m_1} is mapped by ϕ to the set of all triangles of T_{m_2}.

In the case that one of the congruences (2), (4) is satisfied, let $\phi(a_i) = a_{(i-1)(m_2-1)+1}$. The set of all triangles of T_{m_1} is transformed by ϕ to the set

$$\{(a_{(i-1)(m_2-1)+1} a_{i(m_2-1)+1} a_{(i+m_1-1)(m_2-1)+1}) ,$$
$$(a_{(i-1)(m_2-1)+1} a_{i(m_2-1)+1} a_{(v+i-m_1)(m_2-1)+1}) \mid 1 \le i \le v\} .$$

Let $j = 1 + (i-1)(m_2-1)$. We continue exactly as before (with m_2-1 replacing m_2 in the previous calculations).

Example. Let $v = 20$. In T_4 and in T_9, through each vertex, there are three normal circuits with the lengths 20, 20, 5. Therefore they are possibly isomorphic. But an examination of the congruences (1)–(4) in Theorem 2.1 yields that none of these is satisfied by $v = 20$, $m_1 = 4$, $m_2 = 9$, hence $T_4^{20} \not\approx T_9^{20}$. An examination of all the integers m such that $1 \le m \le 10$ [$3 \le m \le 9$] shows that there are exactly eight different regular maps of type $\{3,6\}$ [five different regular triangulations] of the torus with 20 vertices and a normal path through all the vertices. These are obtained for $m = 1,2,3,4,5,6,9,10$ [$m = 3,4,5,6,9$].

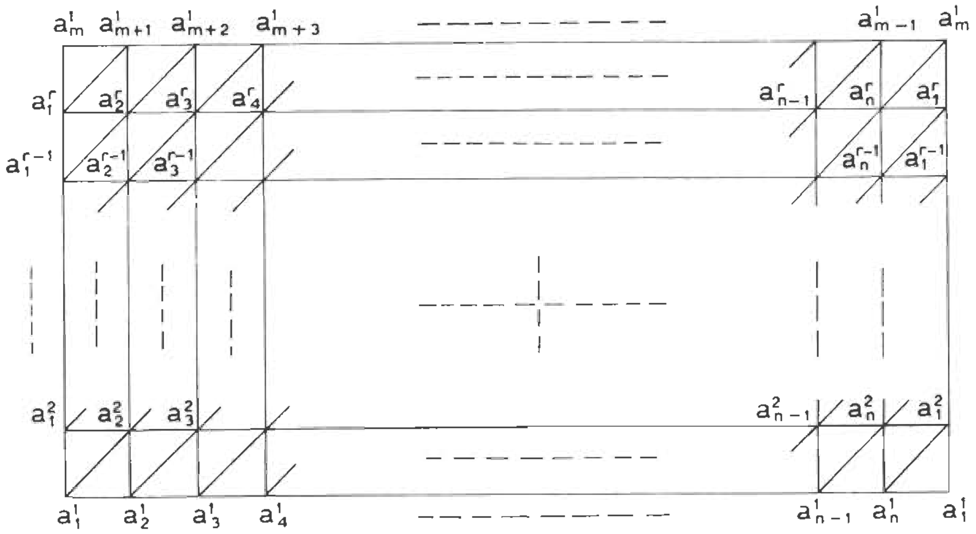

Fig. 2.

3. The general case

Let T be a regular map of type $\{3, 6\}$ on the torus, with v vertices, and let n be the length of the normal circuit with maximal length in T. From [2] and also from the discussion in Section 2, it follows that $r = v/n$ is an integer, and that T can be represented as in Fig. 2 (for $r = 1$ this coincides with Fig. 1). We denote this representation by $T_m^{v,r}$ (or simply by T_m, if there is no danger of mistake). The vertices in $T_m^{v,r}$ are a_i^j for $1 \le i \le n$, $1 \le j \le r$. As for Fig. 1, the vertical sides of the rectangle in Fig. 2 are identified in the natural way, while the identification of the lower and the upper side must be carried out according to the notation of the vertices, and – unless $m = 1$ – needs some shifting.

For $1 < j < r$, the vertex a_i^j is adjacent to the six vertices

$$a_i^{j-1}, a_{i-1}^{j-1}, a_{i-1}^j, a_i^{j+1}, a_{i+1}^{j+1}, a_{i+1}^j;$$

a_i^r is adjacent to

$$a_i^{r-1}, a_{i-1}^{r-1}, a_{i-1}^r, a_{i+m-1}^1, a_{i+m}^1, a_{i+1}^r;$$

and a_i^1 is adjacent to

$$a^r_{i+1-m},\ a^r_{i-m},\ a^1_{i-1},\ a^2_i,\ a^2_{i+1},\ a^1_{i+1}\ .$$

Here and in the following, the lower index is mod n (a^j_0 is a^j_n) and the upper index is mod r (a^0_i is a^r_i).

In $T^{v,r}_m$ for $r > 1$, there are no loops, and double edges exist only if $n = 2$, or $r = 2$ and m equals 1 or n. Hence for $t^{v,r}_m$ to be a triangulation it is sufficient to require that $r > 2$ and if $n = \frac{1}{2}v$ then $m \neq 1, n$.

The normal circuit through $(a^1_1 a^2_1)$ is

$$(a^1_1,\ a^2_1, \ldots, a^r_1,\ a^1_m,\ a^2_m,\ a^1_{l_1(m-1)+1}),$$

where l_1 is minimal with respect to the property $l_1(m-1)+1 \equiv 1$ mod n), i.e., $l_1 = n/(n,m-1)$. Hence the length of this circuit is $n_1 = r \cdot l_1 = v/(n,m-1)$. The normal circuit through $(a^1_1 a^2_2)$ is

$$(a^1_1, a^2_2, \ldots, a^r_r, a^1_{r+m}, a^2_{r+m+1}, \ldots, a^1_{l_2(r+m-1)+1})\ ,$$

where l_2 is minimal with respect to the property $l_2 \cdot (r+m-1)+1 \equiv 1$ (mod n), i.e., $l_2 = n/(n,\ r+m-1)$. Hence the length of the circuit is $n_2 = r \cdot l_2 = v/(n,\ r+m-1)$.

Summarizing, through each vertex of $T^{v,r}_m$ there are three normal circuits of lengths

$$n = \frac{v}{r},\ n_1 = \frac{v}{(n,\ m-1)},\ n_2 = \frac{v}{(n,\ r+m-1)}\ .$$

Note that this holds also for $r = 1$. ($T^{v,1}_m$ is represented in Fig. 1.) As we required that $n_1,\ n_2 \leq n \leq v$, it follows that $(n,\ m-1) \geq r$, $(n,\ r+m-1) \geq r$ and, in particular, $r \leq n = v/r$, i.e. $r^2 \leq v$.

Other simple consequences of this discussion are:

Consequence 3.1. *Let v be prime. In each regular map of type $\{3, 6\}$ on the torus with v vertices, there are three normal circuits, two of which pass through all the vertices and the third either passes through all the vertices or is of length* 1 *(i.e., a loop).*

Consequence 3.2. *Let the three normal circuits through each vertex in a*

$$\chi(v) = \sum_{r|v,\, r^2 \leq v} \nu(v, r) .$$

We investigate the function $\nu(v, r)$ by distinguishing two cases.

Case 1: $r^2 \nmid v$. In this case, $r \nmid n$ and it follows from Theorem 3.1 that $\nu(v, r)$ is the number of integers m such that

$$1 \leq m \leq (v - r^2 + 2r)/2r,\ (n, m-1) > r,\ (n, m+r-1) > r.$$

The last two conditions imply that $m \neq 1, 2, \ldots, r+1$, hence the first condition may be strengthened to

$$r + 2 \leq m \leq (v - r^2 + 2r)/2r$$

from which it follows that

$$\nu(v, r) \begin{cases} = 0 & \text{if } 3r^2 \geq v \text{ and } r^2 \nmid v , \\ \leq \left[\dfrac{v - 3r^2}{2r}\right] & \text{if } 3r^2 < v \text{ and } r^2 \nmid v . \end{cases}$$

This upper bound is far from being the best, since, e.g., for $v = 84$, $r = 3$ its value is 9, while $\nu(84, 3) = 2$. (The two maps $T_m^{84,3}$ are obtained for $m = 5, 8$.)

Case 2: $r^2 \mid v$. In this case, $r \mid n$ and it follows from Theorem 3.3 that

$$\nu(v, r) = \nu_1(v, r) + \nu_2(v, r) ,$$

where $\nu_1(v, r)$ is the number of integers m such that

$$(**) \qquad 1 \leq m \leq (v - r^2 + 2r)/2r,\ r \nmid m-1,\ \ (n, m-1) \geq r,\ \ (n, m+r-1) \geq r,$$

and $\nu_2(v, r)$ is the number of equivalence classes of the integers m such that

$$1 \leq m \leq (v - r^2 + 2r)/2r, \quad r \mid m-1,$$

where $m_1 \neq m_2$ are in the same class if and only if they satisfy one of the congruences (5)–(8). Because of the geometrical nature of the problems, this partition into classes is good, i.e., if m_1, m_2 are in the class A, and m_2, m_3 are in the class B, then $A = B$.

For $\nu_1(v, r)$, the second condition in (**) implies that in the last two conditions there is a strict inequality that holds. Hence, as in case 1, the first condition in (**) may be strengthened:

$$r + 2 \leq m \leq (v - r^2 + 2r)/2r \;.$$

Since the number of the integers m satisfying the last inequality as well as $r \mid m-1$ is at least $[(v-3r^2)/2r^2]$, we get

$$\nu_1(v, r) \begin{cases} = 0 & \text{if } 3r^2 \geq v \text{ and } r^2 \mid v, \\ \leq [(v-3r^2)/2r] - [(v-3r^2)/2r^2] & \text{if } 3r^2 < v \text{ and } r^2 \mid v. \end{cases}$$

For $v = 84$, $r = 2$, this upper bound for $\nu_1(v, r)$ is 9, while $\nu_1(84, 2) = 2$ (and the two maps are obtained for $m = 8, 13$).

Thus, this upper bound for $\nu_1(v, r)$ as well as the bound in the previous case is far from being satisfactory. However, the calculation of $\nu_1(v, r)$ (as well as $\nu(v, r)$ in the previous case), where v and r are given, is not difficult. The calculation of $\nu_2(v, r)$ is more complicated and so it is worthwhile to find bounds for it. This we shall do now.

Assume that r, v, m_1, m_2 are integers such that

$$r^2 \mid v, \quad r \mid m_1 - 1, \quad r \mid m_2 - 1, \quad 1 \leq m_1 \neq m_2 \leq (v - r^2 + 2r)/2r \;.$$

Denote $v/r^2 = s$, $m_1 - 1 = \alpha r$, $m_2 - 1 = \beta r$. Dividing the congruences (5)–(8) by r, we get the congruences (9)–(12) below, and the function $\nu_2(v, r)$ is found to be a function of the one variable s only. Call this function $\lambda(s)$. Therefore, $\lambda(s)$ is the number of the equivalence classes of the integers α, $0 \leq \alpha \leq \frac{1}{2}(s-1)$, such that $\alpha \neq \beta$ which satisfy $0 \leq \alpha, \beta \leq \frac{1}{2}(s-1)$ are in the same class if and only if one of the following congruences (9)–(12) is satisfied.

(9) $\alpha + \beta + \alpha\beta \equiv 0 \pmod{s}$,
(10) $\alpha\beta + \beta + 1 \equiv 0 \pmod{s}$,
(11) $\alpha\beta + \alpha + 1 \equiv 0 \pmod{s}$,
(12) $\alpha\beta \equiv 1 \pmod{s}$.

regular map of type $\{3,6\}$ *on the torus with* v *vertices be* n, n_1, n_2 *with* $n_1, n_2 \leq n$. *Then* $n \cdot n_1 \geq v$, $n \cdot n_2 \geq v$.

The above discussion together with Theorem 2.1 form the proof of a part of the next theorem. However, we omit the proof of the remaining part, i.e., that $1 \leq m \leq \frac{1}{2}(n-r+2)$, as well as the proof of Theorem 3.3, since both proofs (in particular that of Theorem 3.3) yield very long calculations. The proof are available from the author.

Theorem 3.1. *Let* T *be a regular map of type* $\{3,6\}$ *on the torus with* v *vertices, and let* n *be the length of the normal circuit in* T *with maximal length. Then* $n|v$ *and there is an integer* m *such that* $T \approx T_m^{v,r}$, *where* $r = v/n$, $1 \leq m \leq \frac{1}{2}(n-r+2)$, $(n, m-1) \geq r$, $(n, r+m-1 \geq r$; *and the three normal circuits through each vertex are of lengths*

$$n = \frac{v}{r}, n_1 = \frac{v}{(n, m-1)}, n_2 = \frac{v}{(n, r+m-1)}.$$

It follows from Theorem 3.1 that if $m \neq 1$ then $r \leq m-1$, hence

$$r \leq \frac{1}{2}(n-r) = \frac{v-r^2}{2r},$$

i.e., $3r^2 \leq v$.

The analogous theorem for regular triangulations is the suitable part of Theorem 2.1 together with:

Theorem 3.2. *Let* T *be a regular triangulation of the torus with* v *vertices, in which the normal circuit of maximal length is of length* $n < v$. *Then* $n \neq 2$, $n|v$, *and there exists an integer* m *such that* $T \approx T_m^{v,r}$ *where* $r = v/n$, $1 \leq m \leq \frac{1}{2}(n-r+2)$ (*and if* $n = \frac{1}{2}v$ *then* $r+1 \leq m \leq \frac{1}{2}(n-r+2)$, $(n, m-1) \geq r$, $(n, r+m-1) \geq r$. *The lengths of the normal circuits through each vertex are as in Theorem* 3.1.

The following question arises: Let v, n, r be as in Theorem 3.1, and let $1 \leq m_i \leq \frac{1}{2}(n-r+2)$, $(n, m_i-1) \geq r$, $(n, r+m_i-1) \geq r$ $(i = 1, 2)$. When does $T_{m_1} \approx T_{m_2}$? Clearly for this to hold it is necessary that the numbers (n, m_1-1), $(n, r+m_1-1)$ be equal (not necessarily in correspondence) to the numbers (n, m_2-1), $(n, r+m_2-1)$. But is this condition also sufficient? The next theorem deals with this question.

Theorem 3.3. *In the notation of Theorem* 3.1, *for* $m_1 \neq m_2$ *such that*

$$1 \leq m_i \leq \tfrac{1}{2}(n-r+2),\ (n, m_i-1) \geq r,\ (n, r+m_i-1) \geq r\ (i=1,2),$$

there holds $T^{v,r}_{m_1} \approx T^{v,r}_{m_2}$ *if and only if* $r|n$, $r|m_1-1$, $r|m_2-1$ *and at least one of the following congruences* (5)–(8) *is satisfied*:

(5) $m_1 + m_2 + (m_1-1)(m_2-1)/r \equiv 2 \pmod{n}$,
(6) $r + (r+m_1-1)(m_2-1)/r \equiv 0 \pmod{n}$,
(7) $r + (m_1-1)(r+m_2-1)/r \equiv 0 \pmod{n}$,
(8) $(m_1-1)(m_2-1)/r \equiv r \pmod{n}$.

Moreover, if $T^{v,r}_{m_1} \approx T^{v,r}_{m_2}$ *holds, then at least one of the two inequalities* $(n, m_i-1) \geq r$, $(n, r+m_i-1) \geq r$, *for each i* $(i = 1, 2)$, *is an equality. In other words, in each* $T^{v,r}_{m_i}$ $(i = 1, 2)$ *there are at least two normal circuits of length n through each vertex.*

Because of the notes at the beginning of this section, the theorem analogous to the first part of Theorem 3.3 for regular triangulations of the torus is the suitable part of Theorem 2.1 together with the following.

Theorem 3.4. *Using the notation of Theorem* 3.2, *for* m_1, m_2 *such that* $1 \leq m_1 \neq m_2 \leq (n-r+2)/2$ *(and if* $n = \tfrac{1}{2}v$ *then* $m_1, m_2 \neq 1$), $(n, m_i-1) \geq r$, $(n, r+m_i-1) \geq r$ $(i = 1, 2)$, *an isomorphism between the triangulations* $T^{v,r}_{m_1}$, $T^{v,r}_{m_2}$ *holds if and only if* $r|n$, $r|m_1-1$, $r|m_2-1$ *and at least one of the congruence* (5)–(8) *holds.*

4. The number of regular maps with a given number of vertices

Let $\chi(v)$ denote the number of regular maps of type $\{3, 6\}$ on the torus, with v vertices, which are different from each other up to isomorphism, and let $\nu(v, r)$ be the number of those maps in which the normal circuit of maximal length is of length v/r. Also denote $n = v/r$. By Theorem 3.1, we have

Because of the symmetrical roles of α and β, the congruences (10), (11) are the same; nevertheless, meanwhile we prefer to leave both of them.

For odd $s > 3$, the numbers 1 and $\frac{1}{2}(s-1)$ are in the same class because of (9) (or (10) also), hence nothing is lost if we demand $0 \le \alpha \ne \beta < \frac{1}{2}(s-1)$. But now it can be easily seen that no pair α, β will satisfy more than one of the congruences (9)–(12).

Note that 0 is the only member of its class, and the same with the number 1. For a given α, $2 \le \alpha < \frac{1}{2}(s-1)$, the numbers β which share the same class with α because of (9) are the integers in the set

$$\{(ks-\alpha)/(\alpha+1) \mid 1 \le k \le [\tfrac{1}{2}(\alpha+2)],\ k \text{ is an integer}\},$$

and it can be easily verified that at most one such β exists. In a similar manner, the integers β which share the same class with α because of (10) are the integers in the set

$$\{(ks-1)/(\alpha+1) \mid 1 \le k \le [\tfrac{1}{2}(\alpha+2)],\ k \text{ is an integer}\},$$

and at most one such β exists. Similarly we get that at most one β shares the same class with α because of (11), and at most one β shares the same class with α because of (12).

It follows that if there are l members in a class, then $\binom{l}{2} \le 5$, hence $l \le 3$, i.e., in every class there are at most three members. Therefore we have for $s > 3$

$$\tfrac{1}{2}s \ge \lambda(s) \ge 2 + \tfrac{1}{6}(s-5) = \tfrac{1}{6}(s+7).$$

Sometimes (e.g. for $s = 5, 11, 17, 23$) the equality sign holds in the right side of this inequality, and sometimes (e.g. for $s = 8, 12, 24$) the equality sign holds in the left side (see Table 1).

For $s \le 3$, we have

$$\lambda(1) = \lambda(2) = 1, \quad \lambda(3) = 2.$$

Note that $\lambda(v)$ is exactly the number of the (non-isomorphic) regular maps of type $\{3, 6\}$ on the torus with v vertices and with a normal Hamiltonian circuit. Thus we proved

Table 1
Values of $\lambda(v)$ and $\chi(v)$ $(1 \leq v \leq 24)$

v	$\lambda(v)$	$\chi(v)$	v	$\lambda(v)$	$\chi(v)$
1	1	1	13	4	4
2	1	1	14	5	5
3	2	2	15	6	6
4	2	3	16	6	16
5	2	2	17	4	4
6	3	3	18	7	8
7	3	3	19	5	5
8	4	5	20	8	10
9	3	4	21	8	8
10	4	4	22	7	7
11	3	3	23	5	5
12	6	8	24	12	15

Theorem 4.1. *The number $\lambda(v)$ of regular maps of type $\{3,6\}$ on the torus with exactly v vertices and with a normal Hamiltonian circuit satisfies*

$$\lambda(1)=\lambda(2)=1,\quad \lambda(3)=2,\quad \tfrac{1}{2}v \geq \lambda(v) \geq \tfrac{1}{6}(v+7) \quad (v>3)\,.$$

The following theorem is a simple consequence of Consequence 3.2, of the last part of Theorem 3.3 and of the previous discussion:

Theorem 4.2. *Let v, n_i $(i=1,2,3)$ be positive integers such that $v \geq n_1 \geq n_2 \geq n_3$. There exists a regular map of type $\{3,6\}$ on the torus with v vertices in which the lengths of the normal circuits through each vertex are n_1, n_2, n_3 if and only if $n_i | v$ $(i=1,2,3)$, $n_1 \cdot n_3 \geq v$, and there exists an integer m, $1 \leq m \leq 1+\frac{1}{2}(n_1-(v/n_1))$ such that n_2, n_3 equal (not necessarily correspondingly) the numbers*

$$v/(n_1, m-1), \quad v/\left(n_1, \frac{v}{n_1}+m-1\right).$$

(Then the map is $T_m^{v_1, v/n_1}$.)

Moreover, if $n_1 > n_2$ there exists at most one such a map and if $n_1 = n_2$ at most three such maps exist.

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