f91 indexed

→ 1285
3270
3324

Thue - Sequences. & others

Bibliography (* = main refs)

1. G. A. Hedlund, Remarks on work of Axel Thue on sequences, Nordisk Mat Tidskrift, 15 1967

2. R. C. Entringer et al, On Nonrepetitive sequences, JCT (A) 16 159–164 (1974)

3. Numerology of Dr Matrix, circa p 92. M. Gardner

4. Yaglom & Yaglom, Challenging Problems, 2. pp 12, 204, 94-98, Problem 125 b

5. F. Dejean, Sur un theoreme de Thue, JCT 13(A) 90 '72

6. S.E. Arson, Mat. Sbornik 2 (44) 769-779 1937

7. Gottschalk & Hedlund, Topological Dynamics

8. Dean, A sequence without repeats, ... AMM (April 1965) vol 72 p 383, cited in A3324

Indexed

Examples

1. Sequence N71 = 1285 ✓ T-M. sequence

1221, 2112, 2112, 1221, ...

[rule: 1 → 12 2 → 21]

2. Sequence N155.5 = 3270 [Ref 4 p. 204] ✓

123, 132, 312, 321, 312, ...

[rule: 1 in odd position → 123; in even → 321; 2 (odd) → 231; 2 (even) → 132; 3 (odd) → 312; 3 (even) → 213].

3. Sequence 3324 = N165.5 Ref 8 p383 ie [AMM 72 383 65] ✓

1234, 1432, 1232, 1434, ...

[rule: divide in 4 blocks ABCD. Then ABCD → ABCD ADCB

Sequence

Problem 5030

[AMM 70 675 63]

Does there exist a nonterminating decimal, using digits 1, 2, 3, st no two adjacent n-tuples are identical

Let $S = a_0 a_1 a_1 \cdots$

$= 0110\ 1001\ 1001\ 0110 \cdots$ = Thue sequence

Define $T = b_1 b_2 b_3 \cdots$ where b_i = no of 1's between the i^{th} & $(i+1)^{th}$ occurrence of 0 in S

$= 2102012 \cdots$

= length of runs of 1's, including 0 runs

1283

Nordisk Matematisk Tidsskrift

Vol 15, 1967 [NMT] Indexed

REMARKS ON THE WORK OF AXEL THUE ON SEQUENCES

G. A. HEDLUND

In a series of papers which appeared during the period 1906–1914, Axel Thue attacked and solved difficult combinatorial problems which arise in the study of sequences of finite or infinite length, where the elements are chosen from a finite class S of symbols.

An *unending sequence*

$$x = \ldots x_{-1}x_0x_1x_2\ldots$$

is a function defined on the set of all integers with values in S. The sequence x is *periodic* provided there exists a positive integer w such that $x_{i+w}=x_i$, $i=0, \pm 1, \pm 2, \ldots$. For example, if $S=\{a,b\}$ and we define $x_{2i}=a$, $x_{2i+1}=b$, we obtain the periodic sequence

$$\ldots abababab\ldots.$$

A *block* is a finite sequence of symbols from S. A block B *appears* in x provided there exists an integer i such that $x_{i+1}\ldots x_{i+n}=B$. Thue defines a sequence x as *irreducible* provided no block of the form BB appears in x. In a sense, irreducibility is the antithesis of periodicity. It is not at all obvious that there exist unending sequences which are irreducible.

In the paper [5], Thue constructed unending irreducible sequences, first on four symbols $\{a,b,c,d\}$ and then with the aid of these, he constructed such sequences on three symbols $\{a,b,c\}$.

It is easy to show that in the case of two symbols $\{a,b\}$ there cannot exist unending irreducible sequences. But Thue could show that in this case there exist unending sequences in which no block of the form BBB appears.

In a lengthy paper [6], Thue gave a much more detailed analysis of the irreducibility problem in the cases where the number of symbols is 2 or 3. He defined a sequence on two symbols to be irreducible provided like blocks which appear in it never overlap. Then he essentially determined all irreducible sequences on two symbols and, in particular, constructed the following irreducible unending sequence:

N71

$$(1) \qquad \ldots x(-1),\ x(0),\ x(1),\ x(2), \ldots,$$

where $x(0)=a$, $x(1)=b$, $x(2)x(3)=ba$; $x(2^n)$, $x(2^n+1), \ldots, x(2^{n+1}-1)$ is obtained from $x(0), x(1), \ldots, x(2^n-1)$ by interchanging a and b; and finally, $x(-i)=x(i-1)$ for $i=1,2,3,\ldots$.

In this same paper, Thue constructed large classes of irreducible sequences on three symbols, as well as sequences on n symbols ($n \geqq 4$) which have the strong property that like blocks which appear in them are separated by at least $n-2$ symbols.

Probably due to its publication in a journal with restricted availability, this fundamental work of Thue on sequences has not come to the attention of others who have worked in this area or encountered problems in this connection, and consequently some of his results have been independently rediscovered again and again.

In analyzing the geodesic flow on certain surfaces of negative curvature, Marston Morse [2] showed that a class of unending geodesics on these surfaces could be characterized by unending sequences of symbols, with the geometric properties of a geodesic reflected in the combinatorial properties of the corresponding sequence. In order to show the existence on the surface of an almost periodic (recurrent in the sense of G. D. Birkhoff) geodesic which is not periodic, Morse showed that it was sufficient to construct an unending sequence on two symbols a and b, with the property that it is not periodic and yet almost periodic in the sense that any block appearing in the sequence appears infinitely often to right and left with bounded gap between its appearances. He independently constructed the sequence (1) and showed that it had the desired properties. Although Thue was not concerned with the property of almost periodicity and never defined it, it is interesting to note that in his thorough analysis of irreducible sequences on two symbols, he proved a theorem which has as a consequence the almost periodicity of (1).

Some twenty years later, the problem of showing the possibility of an unending game of chess subject to certain rules as to a forced draw led to the problem of the construction of a sequence on two symbols in which there appears no block of the form BBb, where b is the first element of B, thus precisely the problem which Thue had considered many years previously. Morse, again without knowledge of Thue's work, showed that the unending sequence (1) has the desired property. (See M. Morse and G. A. Hedlund [3].) At the same time the question was raised as to the existence of unending sequences on three symbols in which there appears no block of the form BB (back to Thue again!)

and in the paper cited, such a sequence was constructed. It was one of the many which Thue constructed.

The existence of the highly nonperiodic sequences constructed by Thue have also been found useful in attacks on the unrestricted Burnside problem. The Burnside group $B(n,r)$ is the group with n generators and with relations $g^n=1$ for every $g \in B(n,r)$. The problem is that of determining which of the groups $B(n,r)$ are finite. P. S. Novikoff [4] has announced a proof that if $n \geqq 72$ and $r \geqq 2$, then $B(n,r)$ is infinite. Significant use is made of the existence of infinite sequences on three symbols which do not contain blocks of the form CC and infinite sequences on two symbols which do not contain blocks of the form CCC. Novikoff cites a paper of S. Arŝon [1], who, apparently, independently rediscovered some of Thue's results.

These are but two instances of the rediscovery of Thue's important developments in the structure of sequences. There are numerous others, sometimes where the developments were made for their own interest and others where they were needed in the solution of problems in other areas.

REFERENCES

[1] S. Arŝon: *Proof of the existence of asymmetric infinite sequences.* Recueil Math. (Mat. Sbornik) 2 (1937).

[2] M. Morse: *Recurrent geodesics on a surface of negative curvature.* Trans. Amer. Math. Soc. 22 (1921), pp. 84–100.

[3] M. Morse — G. A. Hedlund: *Unending chess, symbolic dynamics and a problem in semigroups.* Duke Math. J. 11 (1944), pp. 1–7.

[4] P. S. Novikoff: *On periodic groups.* Dokl. Akad. Nauk. SSSR 127 (1959), pp. 749–752.

[5] A. Thue: *Über unendliche Zeichenreihen.* Videnskabsselskabets Skrifter, I Mat.-nat. Kl., Christiania 1906.

[6] A. Thue: *Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. Ibid.*, 1912.

N71 = 1285

MATHEMATISCHES INSTITUT
DER UNIVERSITÄT ZU KÖLN
P. Flor

5 KÖLN 41
WEYERTAL 86-90
RUF: 470 2275

1/3/74

Dr. N.J.A. S l o a n e
Mathematics Research Center
Bell Telephone Laboratories
Murray Hill, N.J. 07974
USA

Sir:

You may be interested in the fact that Sequence No. 71 of your Handbook wich you mention without any reference, is in fact well-known. It is frequently called the "Morse sequence". I can give you the following references :

513.83
G68

W.H. Gottschalk and G.A. Hedlund:
Topological Dynamics (AMS Colloquium Publications, vol. 36, 1955, p. 1o5-113.

S. Arshon (Aršon) Démonstration de l'existence des suites asymétriques infinies
Mathematičeskij sbornik vol. 2 (44), p.779 (1936).

0d
Mond 8
74

According to a recent paper by Hedlund (Nordisk Matematisk Tidsskrift 15, p.148-15o (1967), this sequence was even discussed by Axel Thue, as long ago as 1912 ! At any rate, this particular sequence merits a reference: I should prefer Gottschalk-Hedlund, for wealth of mathematical information, while Hedlund's paper of 1967 (to which I have no access right now) probably gives most of the credit where it belongs. *

Yours sincerely,

Peter Flor

* Remarks on the work of Axel Thue on sequences

N155.5=3270

THE OHIO STATE UNIVERSITY

March 20, 1974

Dr. N.J.A. Sloane
Bell Laboratories
600Mountain Avenue
Murray Hill, New Jersey

Dear Dr. Sloane:

I wish to take advantage of your open invitation to submit sequences. Our earlier correspondence (and the fact that I was able to see your book) put me in the frame of mind in which I was able to recall old friends. One of these is the following:

1, 2, 3, 1, 3, 2, 3, 1, 2, 3, 2, 1, 3, 1, 2, 1 , 3, 2, 3, 1, 2,
3, 2, 1, 2, 3, 1, 2, 1, 3, 2, 3, 1, 3, 2, 1, 3, 1, 2, 3, 2, 1, ...

It is a self-generating sequence with the property that no two consecutive blocks are identical, i.e., for no positive integers p and q can we have $(a_p, a_{p+1}, \ldots, a_{p+q-1}) = (a_{p+q}, a_{p+q+1}, \ldots, a_{p+2q-1})$. I know of two ways to generate this sequence.

One way is found in S.E. Aršon, Matematičeskiĭ Sbornik, vol. 2 (44), pp. 769-779 (1937), and redescribed in A.M. Yaglom and I. M. Yaglom, Challenging mathematical problems with elementary solutions, vol. 2, problem 125b, stated on p. 12, given a hint for on p. 204, and solved on pp. 94-98 (Holden-Day, 1967)(published as Neelementarnye zadači v elementarnom izloženii, Moscow, 1954). Set $s_1 = 123$ (written without commas). If s_k has been defined, let s_{k+1} be the sequence obtained from s_k by replacing each occurrence of 1 in an odd place by 123, each occurrence of 1 in an even place by 321, each occurrence of 2 in an odd place by 231, each occurrence of 2 in an even place by 132, each occurrence of 3 in an odd place by 312, and each occurrence of 3 in an even place by 213. It is easy to see that s_j is an initial subsequence of s_k if $j < k$.

When I first heard of the problem of finding an infinite sequence with terms chosen from {1, 2, 3} but without any consecutive repetitions (long after the problem had been solved), I came up with the same sequence, but was not able to prove that it had the required property of non-repetition. Set $s_0 = 1$, and set $s_{k+1} = s_k s_k^{*} s_k^{**}$, where the star indicates the operation of reversing the order of the terms of the sequence and increasing each by 1 (modulo 3). Thus, $s_1 = 1\ 2\ 3$, $s_2 = 123\ 132\ 312$, ...

A bibliography of sequences related to this one can be found in: Françoise Dejean, Sur un théorème de Thue, Journal of Combinatorial

Department of Mathematics / 231 West 18th Avenue / Columbus, Ohio 43210 / Phone (614) 422-4975

Theory, vol. 13, section A, pp. 90-99 (1972).

I was amused by your list of sequences not to be included in your m2m. list. One was similar to one I had heard as a child in Brooklyn: 14, 18, 23, 28, 34, ... Your specification of it ~~these~~ was incomplete, since these are the numbers of the streets after which the stations of the West Side IRT subway are named; the station numbers are different on the East side. This sequence can be quibbled with. The station you considered to represent 34 really has merely a big sign PENNSYLVANIA on it (not even the more correct Pennsylvania Station), and was formerly thought of as representing 33 St. It is probable that certain other stations use numbers only secondarily (42, 59, 110, 116).* A station closed down in the past 20 years or so is 91 St. The stations north of Times Square were part of the original subway line in New York (opened in 1904, much later than Sloane Square and South Kensington Stations of Gilbert and Sullivan fame).

May your tribe increase!

No reply is needed.

Cordially,

Roy Meyers

Leroy F. Meyers

* and Dyckman St can be considered to be the same as 200 st, since the station a block away, on the A line, is called 200th Dyckman St.

Lit^y Check: (from Di Matrix)

*1 AMM Braunholtz 70 1963 675-676
— Seq!

2. BAMS 44 (1938) p 632 Morse

3. Hawkins +, Math. Stud. 24 (1956) pp 185-7

4. Hedlund +, PAMS 15 1964 70-74

√5. Dean AMM 72 (1965) 383-385 √
⇒ Seq 3324 = N165.5

6 DMJ. 11 (1944) 1-7 Morse & Hedlund

7 Leech, Math Gazette 41 (1957) 277-8

In *Mathematics Magazine*, January 1951, Dewey Duncan defined a "heterosquare" as one in which no two rows, columns or diagonals (including "broken diagonals") have the same sum. (The order-3 square has four broken diagonals. Referring to the square shown in Figure 2, they are the cells bearing the following number triplets: 1,6,4; 8,2,5; 3,8,6; and 2,4,7.) Duncan asked for a heterosquare of order-3 and proof that no such square of order-2 exists. It is easy to show that an order-2 is impossible. A proof that the order-3 also is impossible was given by Charles F. Pinzka in *Mathematics Magazine*, September-October 1965, pages 250–252. Order-4 squares *are* possible; Pinzka gave two. Another proof of impossibility for the order-3 was given by Prasert Na Nagara in the same magazine, September-October 1966, pages 255–256. Nagara also found two "almost" heterosquares in which all sums but two were distinct.

J. A. Lindon, writing in *Recreational Mathematics Magazine*, February 1962, proposed searching for antimagic squares in which the sums of the rows, columns and main diagonals (broken diagonals not considered) are not only different but also form a sequence of consecutive integers. A summary of Lindon's results, with some new material added, appears in Joseph Madachy's *Mathematics on Vacation* (Scribner, 1966), pages 101–110. No order-2 square of this type is possible. It is believed that the order-3 also is impossible, although one can come close, as the following square (from C. C. Verbeek's *Puzzel Met Plezier*, Amsterdam, 1962, page 155) shows:

268
791
534

All eight sums are distinct, and only one diagonal sum, 22, is outside the sequence.

Many order-4 antimagic squares, with all sums in consecutive order, were found by Lindon. The number of distinct order-3 antimagic squares (sums not in consecutive order) is not yet known.

2. The equation asked for in the second problem is:

$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$

I am indebted to Russell L. Linton, Oakland, California, for pointing out in a letter that the first integer in the series of such equations is obtained by the formula $n(2n+1)$, where n is the number of terms on the right side of the equation. Thus, to write the next example, which has five terms on the right, we substitute 5 for n to obtain: $5(10+1) = 55$. We can immediately write:

$$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 \\ = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$$

A discussion of this series, "Runs of Squares," by T. H. Beldon, appeared in *The Mathematical Gazette*, December 1961, pages 334–335.

The series has a trivial analog with the following first-power series:

$$1 + 2 = 3$$
$$4 + 5 + 6 = 7 + 8$$
$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

3. The three-symbol chain problem has a fascinating history that begins with a two-symbol chain first discovered by the Norwegian mathematician Axel Thue and described by him in 1912. Begin with 01. For the 0, substitute 01, and for the 1, substitute 10. The result is a chain of four digits: 0110. Repeating this procedure, changing each 0 to 01 and each 1 to 10, produces the chain: 01101001. In this way we can form a chain as long as we wish, each step doubling the number of digits and forming a

chain that starts by repeating the previous chain. This sequence of symbols, called the Thue series, has the remarkable property that no block of one or more digits ever appears three times consecutively. The chain may "stutter" once, but whenever this occurs, regardless of the size of the block that repeats, the very next digit is sure to be the wrong one for a third appearance of the block.

Max Euwe, a former world chess champion, was among the first to recognize that the Thue sequence provides a method of playing an infinite game of chess. The so-called "German rule" for preventing such games declares a game drawn if a player plays any finite sequence of moves three times in succession. Two players need only create a position in which each can move either of two pieces back and forth, regardless of how the other player moves his two pieces. If each now plays his two pieces in a Thue sequence, neither will ever repeat a pattern of moves three times consecutively.

From the Thue series it is easy to derive a three-symbol chain that solves Dr. Matrix' problem. First, we transform it to a chain of four symbols by writing 0 under every 00 pair, 1 under every 01 pair, 2 under every 10 pair, and 3 under every 11 pair:

Thue series:	0 1 1 0 1 0 0 1 . . .
Four-symbol chain:	1 3 2 1 2 0 1 . . .

This infinite four-symbol chain has the property that no finite block of digits ever appears *twice* side by side. It can now be transformed to a three-symbol chain, with the same property, by replacing every 3 with a 0:

Four-symbol chain:	1 3 2 1 2 0 1 . . .
Three-symbol chain:	1 0 2 1 2 0 1 . . .

This solution to the three-symbol problem was given by Marston Morse and Gustav Hedlund in an important 1944 paper,

"Unending Chess, Symbolic Dynamics and a Problem in Semi-groups," *Duke Mathematics Journal*, Vol. 11 (1944), pages 1–7. There were earlier solutions (including one by the Russian mathematician S. Arshon in 1937) and many later ones. John Leech gave this solution in "A Problem on Strings of Beads," *Mathematical Gazette*, Vol. 41 (1957), pages 277–278:

Consider the following three blocks of digits:

0 1 2 1 0 2 1 2 0 1 2 1 0

1 2 0 2 1 0 2 0 1 2 0 2 1

2 0 1 0 2 1 0 1 2 0 1 0 2

The digits in these blocks are so arranged that if we substitute the three blocks for the three digits (replacing 1 with one block, 2 with another, 3 with the third) in any stutter-free chain (*e.g.*, any one of the three blocks), the resulting chain will also be stutter-free. In this longer chain we can now substitute blocks for digits once more to obtain a still longer chain, and so on ad infinitum.

It is not possible to construct shorter palindromic blocks (blocks that are the same backward as forward) that can be used in this way, but shorter asymmetric blocks are possible. Allan Beek sent me a similar solution using the following asymmetric blocks of eleven digits each:

12313231213
12321312132
12321323132

It is not known if there is a set of three shorter blocks that provides a proof of this type.

The three-symbol chain furnishes a way of evading the rule for drawn chess games even if the rule is strengthened by declaring a game drawn if a finite sequence of moves occurs only twice in succession. Each player simply moves three pieces in a pattern given by the three-symbol chain.

There are other ways of generating the Thue series than the one explained above. Dana Scott, in 1961, sent me the following. First write the sequence of integers in binary form: 0, 1, 10, 11, 100, 101, 110, 111, 1000 . . . Next, replace each number with 1 if it contains an odd number of 1's, and with 0 if it contains an even number of 1's. The result, surprisingly, is the Thue series: 011010011 . . .

A method of transforming the Thue series directly to a three-symbol solution of Dr. Matrix' problem was explained in 1963 by C. H. Braunholtz, "An Infinite Sequence of 3 Symbols with no Adjacent Repeats," *American Mathematical Monthly*, Vol. 70 (1963), pages 675–676. In the Thue series the number of 1's between any 0 and the next 0 is either 0, 1 or 2. There are two 1's between the first and second 0, one 1 between the second and third 0's, none between the third and fourth, and so on. The numbers of these 1's, as we proceed from 0 to 0, form a three-symbol infinite series, 2102012 . . . , with the required property.

P. Erdös proposed the following three-symbol chain problem that is the same as the one given by Dr. Matrix except that two blocks of digits are now considered "identical" if each symbol appears in them the same number of times. For example, 00122 = 02102 because each contains two 0's, one 1, and two 2's. The largest possible sequence that does not have two "identical" blocks side by side is one of seven digits; *e.g.*, 0102010. It is not yet known if there is an infinite four-symbol chain with this property.

Other references on the Thue series and the three-symbol problem include:

Marston Morse, "A Solution of the Problem of Infinite Play in Chess," Abstract 360, *Bulletin of the American Mathematical Society*, Vol. 44 (1938), page 632.

D. Hawkins and W. E. Mientka, "On Sequences which Contain no Repetitions," *Mathematics Student*, Vol. 24 (1956), pages 185–187.

G. A. Hedlund and W. H. Gottschalk, "A Characterization of the Morse Minimal Set," *Proceedings of the American Mathematical Society*, Vol. 15 (1964), pages 70–74.

Richard A. Dean, "A Sequence without Repeats," *American Mathematical Monthly*, Vol. 72 (1965), pages 383–385.

4. The number 102564 quadruples in size if the 4 is moved from the back to the front, 410256; therefore Miss Toshiyori's telephone number is 1-0256. Puzzles of this type are easily solved by a kind of multiply-as-you-go technique explained in Figure 12.

After mastering this method, readers may wish to tackle the following three problems:

1. What is the smallest number ending in 6 that becomes six times as large when the 6 is shifted from the end to the front? (Warning: The number has 58 digits!)

2. Find the smallest number *beginning* with 2 that triples when the 2 is moved to the end.

3. Prove that there is no number beginning with the digit n that increases n times when the first digit is moved from the front to the end, except in the trivial case where n is 1.

Readers interested in further explorations of problems of this type, in which digits are moved from one end of a number to another to accomplish specified results, will find helpful the following references:

Aaron Bakst, *Mathematical Puzzles and Pastimes* (Van Nostrand, 1954), page 177f.

L. A. Graham, *Ingenious Mathematical Problems and Methods* (Dover, 1959), Problem No. 72.

Dan Pedoe, *The Gentle Art of Mathematics* (Macmillan, 1958), page 11f.

W. B. Chadwick, "On Placing the Last Digit First," *American Mathematical Monthly*, Vol. 48 (1941), page 251.