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Will send numerous sequences soon after my return from Minneapolis.

Best — R

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A Class of Binomial Sums and a Series Transform

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Abstract. This paper has been motivated by a study of binomial sums of the type

(1) $$g(n) = \sum_{k=0}^{n} \binom{n+k}{k} f(k)$$

with inverse of the form

(2) $$\frac{(2n)!}{n!} f(n) = \sum_{k=0}^{n} A_k^n g(k)$$

We sum special cases of (1) and develop the special properties of A_k^n and then generalize to consider briefly the series transform

(3) $$g(n) = \sum_{k=0}^{n} \binom{a+bk+n}{k} f(k)$$

and its inverse in the form

(4) $$f(n) \prod_{j=1}^{n} \binom{a+bj+j}{j} = \sum_{k=0}^{n} B_k^n(a,b) g(k)$$

This transform and its inverse are not available in the previously published papers of the author. Effective recurrence relations for computation of the coefficients are given along with tables of values.

1. Introduction

This paper has been motivated by an enquiry from John Wrench [9] who asked me how to prove that

$$S_n = \sum_{k=0}^{n} \binom{n+k}{k} 2^{-k} = 2^n, \qquad n \geq 0. \tag{1.1}$$

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We discuss this and some variations and then generalize to consider the series transform

$$g(n) = \sum_{k=0}^{n} \binom{a+bk+n}{k} f(k) \tag{1.2}$$

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and its inver[illegible] the form

$$f(n)\prod_{j=1}^{n}\binom{a+bj+j}{j}=\sum_{k=0}^{n}B_k^n(a,b)g(k) \tag{1.3}$$

This transform and its inverse are not available in my published papers concerning inverse series pairs. See [3] for summary of most of my papers that would be related to this. Also, (1.2)–(1.33) is not given by Riordan [7]. We concentrate in some detail on the case where $a=0$, $b=1$.

2. Summation of Elementary Series

To prove (1.1) we will derive the recurrence relation

$$S_{n+1}=2S_n \tag{2.1}$$

Then since $S_0=1$, the desired formula follows at once by mathematical induction. The proof of (2.1) may be done as follows:

$$\begin{aligned}
S_{n+1}&=\sum_{k=0}^{n+1}\binom{n+1+k}{k}2^{-k}=\sum_{k=0}^{n+1}\left\{\binom{n+k}{k}+\binom{n+k}{k-1}\right\}2^{-k}\\
&=\sum_{k=0}^{n+1}\binom{n+k}{k}2^{-k}+\sum_{k=1}^{n+1}\binom{n+k}{k-1}2^{-k}\\
&=\sum_{k=0}^{n}\binom{n+k}{k}2^{-k}+\binom{2n+1}{n+1}2^{-n-1}+\sum_{k=0}^{n}\binom{n+1+k}{k}2^{-k-1}\\
&=S_n+\binom{2n+1}{n+1}2^{-n-1}+\frac{1}{2}\sum_{k=0}^{n}\binom{n+1+k}{k}2^{-k}\\
&=S_n+\binom{2n+1}{n+1}2^{-n-1}+\frac{1}{2}\left\{\sum_{k=0}^{n+1}\binom{n+1+k}{k}2^{-k}\right.\\
&\qquad\left.-\binom{2n+2}{n+1}2^{-n-1}\right\}\\
&=S_n+\binom{2n+1}{n+1}2^{-n-1}+\frac{1}{2}S_{n+1}-\binom{2n+2}{n+1}2^{-n-2}
\end{aligned}$$

whence

$$\frac{1}{2}S_{n+1}-S_n=\binom{2n+1}{n+1}2^{-n-1}-\binom{2n+2}{n+1}2^{-n-2}=0.$$

Thus we have proved that $S_{n+1}=2S_n$ as desired.

For six references to (1.1) in the literature, see Hansen [4, (7.1.10)].

We may use the same technique to prove that

$$T_n=\sum_{k=0}^{\infty}\binom{n+k}{k}2^{-k}=2^{n+1},\qquad n\geq 0 \tag{2.2}$$

Indeed

$$\begin{aligned}
T_{n+1}&=\sum_{k=0}^{\infty}\binom{n+1+k}{k}2^{-k}\\
&=\sum_{k=0}^{\infty}\left\{\binom{n+k}{k}+\binom{n+k}{k-1}\right\}2^{-k}\\
&=\sum_{k=0}^{\infty}\binom{n+k}{k}2^{-k}+\sum_{k=1}^{\infty}\binom{n+k}{k-1}2^{-k}\\
&=T_n+\sum_{k=0}^{\infty}\binom{n+1+k}{k}2^{-k-1}\\
&=T_n+\frac{1}{2}\sum_{k=0}^{\infty}\binom{n+1+k}{k}2^{-k}\\
&=T_n+\frac{1}{2}T_{n+1}
\end{aligned}$$

whence we have proved that

$$T_{n+1}=2T_n. \tag{2.3}$$

Since $T_0=\sum_{k=0}^{\infty}2^{-k}=2$, then $T_1=4$, etc. and in general by induction $T_n=2^{n+1}$, $n\geq 0$.

Note that the two sums S_n and T_n both satisfy the same recurrence relation, i.e. $f(n+1)=2f(n)$.

This may suggest that we may show that

$$U_n(x)=\sum_{k=0}^{\infty}\binom{n+k}{k}x^k=(1-x)^{-n-1},\qquad |x|<1,$$

by the same technique.

Indeed, virtually the same steps we just used suffice to show that

$$U_{n+1}(x)=U_n(x)+xU_{n+1}(x),$$

which yields

$$U_{n+1}(x) = (1-x)^{-1} U_n(x). \tag{2.5}$$

Since $U_0(x) = \sum_{k=0}^{\infty} x^k = (1-x)^{-1}, |x| < 1$, then (2.4) follows by induction.
Of course, since

$$\binom{n+k}{k} = (-1)^k \binom{-n-1}{k}, \tag{2.6}$$

it follows that we can evaluate (2.4) directly from the binomial theorem as follows:

$$\begin{aligned} U_n(x) &= \sum_{k=0}^{\infty} \binom{n+k}{k} x^k = \sum_{k=0}^{\infty} (-1)^k \binom{-n-1}{k} x^k \\ &= \sum_{k=0}^{\infty} \binom{-n-1}{k} (-x)^k = (1-x)^{-n-1}, \qquad |x| < 1. \end{aligned}$$

Next, let us be more ambitious and try to use the same technique to find a sum for the finite series

$$W_n(x) = \sum_{k=0}^{n} \binom{n+k}{k} x^k, \qquad n \geq 0. \tag{2.7}$$

We find readily the recurrence relation

$$(1-x) W_{n+1}(x) = W_n(x) + \binom{2n+1}{n+1} (1-2x) x^{n+1}, \quad n \geq 0. \tag{2.8}$$

Using this together with the fact that $W_0 = 1$, we may calculate as many Ws as desired, but the recurrence (finite difference equation) does not have a simple closed solution form of the sort we found before for S_n, T_n, or $U_n(x)$.

Let us next endeavor to find a generating function for $W_n(x)$. We define

$$\mathcal{W}(x,t) = \sum_{n=0}^{\infty} W_n(x) t^n. \tag{2.9}$$

Then

$$\begin{aligned} \mathcal{W}(x,t) &= 1 + \sum_{n=1}^{\infty} W_n(x) t^n = 1 + t \sum_{n=1}^{\infty} W_n(x) t^{n-1} \\ &= 1 + t \sum_{n=0}^{\infty} W_{n+1}(x) t^n \end{aligned}$$

Therefore

$$\begin{aligned} (1-x)\mathcal{W}(x,t) &= 1 - x + t \sum_{n=0}^{\infty} (1-x) W_{n+1}(x) t^n \\ &= 1 - x + t \sum_{n=0}^{\infty} \left\{ W_n(x) + \binom{2n+1}{n} (1-2x) x^{n+1} \right\} t^n \\ &= 1 - x + t \sum_{n=0}^{\infty} W_n(x) t^n + tx(1-2x) \sum_{n=0}^{\infty} \binom{2n+1}{n} (tx)^n \\ &= 1 - x + t\mathcal{W}(x,t) + (1-2x) \sum_{n=0}^{\infty} \binom{2n+1}{n} (tx)^{n+1}. \end{aligned}$$

Thus we have

$$(1-x-t)\mathcal{W}(x,t) = 1 - x + (1-2x) \sum_{n=0}^{\infty} \binom{2n+1}{n} (tx)^{n+1}. \tag{2.10}$$

Since it is readily shown that

$$\sum_{n=0}^{\infty} \binom{2n+1}{n} z^n = \frac{(1-4z)^{-1/2} - 1}{2z}, \quad |z| < 1/4, \tag{2.11}$$

then we find

$$\begin{aligned} (1-x-t)\mathcal{W}(x,t) &= 1 - x + (1-2x) \frac{(1-4x)^{-1/2} - 1}{2} \\ &= \frac{1}{2} \left\{ 1 + (1-2x)(1-4tx)^{-1/2} \right\} \end{aligned}$$

so that we have a formula for the generating function $\mathcal{W}(x,t)$:

$$\mathcal{W}(x,t) = \frac{1 + (1-2x)(1-4tx)^{-1/2}}{2(1-x-t)}. \tag{2.12}$$

Thus $W_n(x)$ may be found as the coefficient of t^n in the formal power series expansion of (2.12). As a partial check of (2.12), note that when $x = 1/2$ we get

$$\mathcal{W}(1/2, t) = \frac{1}{1-2t} = \sum_{n=0}^{\infty} (2t)^n = \sum_{n=0}^{\infty} W_n(x) t^n$$

which gives $W_n(1/2) = 2^n$ in agreement with our original formula (1.1). Also $\mathcal{W}(0,t) = \frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$ so that $W_n(0) = 1$ as is clear.

Finally, note that $\mathcal{W}(1,t) = \frac{(1-4t)^{-1/2}-1}{2t} = \sum_{n=0}^{\infty}\binom{2n+1}{n}t^n$ by (2.11) so that $W_n(1) = \binom{2n+1}{n}$, which gives the known binomial identity

$$\sum_{k=0}^{n}\binom{n+k}{k} = \binom{2n+1}{n}. \tag{2.13}$$

Theorem (Gould [1, (1.78)]).

$$\sum_{k=0}^{n}\binom{n+k}{k}\{(1-x)^{n+1}x^k + x^{n+1}(1-x)^k\} = 1. \tag{2.14}$$

The proof is by extensive series manipulation.

If we replace x by $1/x$ this can be stated in the alternative equivalent form

$$\sum_{k=0}^{n}\binom{n+k}{k}\frac{(x-1)^{n+1}-(x-1)^k}{x^k} = x^{n+1}. \tag{2.15}$$

In view of our definition (2.9), we may rewrite (2.14) as

$$(1-x)^{n+1}W_n(x) + x^{n+1}W_n(1-x) = 1. \tag{2.16}$$

Theorem.

$$(1-2t)\mathcal{W}(x,t) - (1-2x)\mathcal{W}(t,x) = \frac{x-t}{1-x-t}. \tag{2.17}$$

Proof. Recall (2.12) and note that the generating function is almost symmetric in x and t. More precisely

$$(1-2t)\mathcal{W}(x,t) - \frac{1-2t}{2(1-x-t)}$$

is symmetric in x and t. Relation (2.17) follows easily from this.

Theorem.

$$x\sum_{k=1}^{n}W_k(x) = W_n(x) - 1 + (2x-1)\sum_{k=0}^{n-1}\binom{2k+1}{k}x^{k+1}. \tag{2.18}$$

Proof. From (2.8) we have

$$W_{k+1}(x) - W_k(x) = xW_{k+1}(x) + (1-2x)x^{k+1}\binom{2k+1}{k}.$$

Sum both sides from $k=0$ to $k=n-1$, and we obtain (2.18) a[illegible]e.

Relation (2.18) allows us to write

$$x\sum_{k=1}^{n}W_k(x) - W_n(x) = x(2x-1)\sum_{k=0}^{n-1}\binom{2k+1}{k}x^k - 1$$

so that by letting $n\to\infty$ and using (2.11) we get

$$\lim_{n\to\infty}\left\{x\sum_{k=0}^{n}W_k(x) - W_n(x)\right\} = \frac{(2x-1)(1-4x)^{-1/2}-3}{2}. \tag{2.19}$$

By manipulating the binomial series [2, Vol II, pp.234–235] it is not difficult to show that

$$W_n(x) = \frac{1}{(1-x)^{n+1}} + \binom{2n}{n}x^n - \sum_{k=0}^{n}\binom{2n}{k+n}\frac{x^{k+n}}{(1-x)^{k+1}} \tag{2.20}$$

3. A Special Binomial Series Transform and its Inverse

We now consider a series transform in a slightly more general setting. We define

$$g(n) = \sum_{k=0}^{n}\binom{n+k}{k}f(k) \tag{3.1}$$

Here are some examples of this transform:

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$$\begin{aligned}
g(0) &= f(0),\\
g(1) &= f(0) + 2f(1),\\
g(2) &= f(0) + 3f(1) + 6f(2)\\
g(3) &= f(0) + 4f(1) + 10f(2) + 20f(3),\\
g(4) &= f(0) + 5f(1) + 15f(2) + 35f(3) + 70f(4),\\
g(5) &= f(0) + 6f(1) + 21f(2) + 56f(3) + 126f(4) + 252f(5),\\
g(6) &= f(0) + 7f(1) + 28f(2) + 84f(3) + 210f(4) + 462f(5) + 924f(6),\\
&\dots
\end{aligned}$$

It is easily seen that an inverse exists and is unique, since we have a non-singular linear transformation. By direct computations, the reader may verify the following examples of the inverse:

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$$\begin{aligned}
f(0) &= g(0),\\
2f(1) &= g(1) - g(0),\\
12f(2) &= 2g(2) - 3g(1) + g(0),\\
120f(3) &= 6g(3) - 10g(2) + 3g(1) + g(0),\\
1680f(4) &= 24g(4) - 42g(3) + 10g(2) + 9g(1) - g(0),\\
30240f(5) &= 120g(5) - 216g(4) + 42g(3) + 50g(2) + 21g(1) - 17g(0),\\
665280f(6) &= 720g(6) - 1320g(5) + 216g(4) + 294g(3) + 230g(2) - 33g(1) - 107g(0),\\
&\dots
\end{aligned}$$

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The inve[illegible]ay be written conveniently in the form

$$\frac{(2n)!}{n!} f(n) = \sum_{k=0}^{n} A_k^n g(k), \tag{3.2}$$

where the coefficients A_k^n are independent of f and g. The inverse is also non-singular so (3.1) and (3.2) imply each other.

Except for obvious features (like e.g. $A_n^n = n!$) the sequences which appear along rows and diagonals are not listed in Sloane's extensive index of sequences [8], and so presumably little, if anything, has been said about these coefficients in the literature.

The inverse pair (3.1)–(3.2) may be thought of in terms of matrices, one matrix being the inverse of the other. For example, with $n = 3$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 6 & 0 \\ 1 & 4 & 10 & 20 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 1/12 & -3/12 & 2/12 & 0 \\ 1/120 & 3/120 & -10/120 & 6/120 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finding the inverse (3.2) of transform (3.1) is equivalent to inverting the matrix $A = (a_{ij})$, where $a_{ij} = \binom{i+j}{i}$ when $0 \le i, j \le n$ and $a_{ij} = 0$ whenever $j > i$.

Theorem.

$$g(n) = 1, \quad n \ge 0, \quad \text{implies } f(n) = \delta_0^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \ge 1 \end{cases}. \tag{3.3}$$

Proof. This can be seen when we set $g(n) = 1$ and compute $f(n)$ directly by solving (3.1) step by step to determine $f(n)$. An immediate consequence of this is that the As in any row sum to zero. That is, we have the

Theorem.

$$\sum_{k=0}^{n} A_k^n = \delta_0^n. \tag{3.4}$$

Proof. Set $g(k) = 1, k \ge 0$, in (3.2) and apply (3.3).

Recurrence relations for commmputation of the A's:

$$\sum_{k=0}^{j} \binom{j+k}{k} A_k^n = \frac{(2n)!}{n!} \delta_j^n, \quad 0 \le j \le n \tag{3.5}$$

which allows a check across any row, and

$$\sum_{k=j}^{n} \binom{n+k}{k} \frac{k!}{(2k)!} A_j^k = \delta_j^n, \quad 0 \le j \le n. \tag{3.6}$$

which allows a check along any diagonal.

Relation (3.5) is obtained by substituting $g(n)$ from (3.1) into (3.2) and noting that all terms must vanish except the one that gives $f(n)$. Similarly we may substitute the value of $f(n)$ from (3.2) into (3.1) and obtain (3.6).

When $j = 0$ in (3.5) we have an immediate proof of (3.4). The most effective form of (3.6) is

$$A_j^n = -\sum_{k=j}^{n-1} \frac{(n+k)!}{(2k)!} A_j^k \tag{3.7}$$

We use this for a fixed n and successive values of j to find row n and check it by using (3.4).

Some other special case relations involving the A's are as follows:

$$A_n^n = n!, \quad n \ge 0, \tag{3.8.1}$$

$$A_{n-1}^n = -(2n-1)A_{n-1}^{n-1} = -(2n-1)(n-1)!, \quad n \ge 1, \tag{3.8.2}$$

$$A_{n-2}^n = -A_{n-2}^{n-1} = (2n-3)(n-2)!, \quad n \ge 2, \tag{3.8.3}$$

$$A_{n-3}^n = (2n-5)A_{n-3}^{n-1} = (2n-5)^2(n-3)!, \quad n \ge 3, \tag{3.8.4}$$

$$A_{n-4}^n = (2n-8)A_{n-4}^{n-1} + (2n-9)A_{n-4}^{n-2}, \quad n \ge 4, \tag{3.8.5}$$

The last relation appears neater when written as

$$A_n^{n+4} = (2n)A_n^{n+3} + (2n-1)A_n^{n+2}, \quad \text{for } n \ge 0, \tag{3.8.6}$$

We can offer other *special identities* involving the A_k^n which we find from *special binomial identities.* Our original motivating formula (1.1) is the case $f(n) = 2^{-n}$, $g(n) = 2^n$. Applying the inverse (3.2) we find at once that

$$\sum_{k=0}^{n} A_k^n 2^k = \frac{(2n)!}{n!\, 2^n} = \prod_{k=1}^{n} (2k-1) \tag{3.9}$$

so that this sum is equal to the product of the first n odd numbers.

Another special result follows by inversion of (2.13). Here $f(n) = 1$ and $g(n) = \binom{2n+1}{n}$. Using the inverse (3.2) we find now that

$$\sum_{k=0}^{n} A_k^n \binom{2k+1}{k} = \frac{(2n)!}{n!} = 2^n \prod_{k=1}^{n} (2k-1) \tag{3.10}$$

A curiou[illegible]ntity called to my attention by Donald Knuth [6], [1, (3.155)]

$$\sum_{k=0}^{s-1} \binom{k}{j}\binom{k+m}{m} = \binom{s}{j}\binom{s+m}{m}\frac{s-j}{m+j+1} \tag{3.11}$$

gives the special case (when $s = n+1, m = n$)

$$\sum_{k=0}^{n} \binom{n+k}{k}\binom{k}{j} = \binom{n+1}{j}\binom{2n+1}{n}\frac{n-j+1}{n+j+1} \tag{3.12}$$

which includes (2.13) as the special instance when $j = 0$. Since $\binom{k}{j}$ is independent of n, we may invert (3.12) by (3.2) and obtain

$$\sum_{k=0}^{n} A_k^n \binom{k+1}{j}\binom{2k+1}{k}\frac{k-j+1}{k+j+1} = \binom{n}{j}\frac{(2n)!}{n!}, \tag{3.13}$$

which gives a kind of expansion of the binomial coefficient $\binom{n}{j}$ in terms of the A_k^n coefficients.

4. The General Binomial Transform and its Inverse

For the more general binomial series transform (1.2)

$$g(n) = \sum_{k=0}^{n} \binom{a+bk+n}{k} f(k)$$

and its inverse (1.3)

$$f(n)\prod_{j=1}^{n}\binom{a+bj+j}{j} = \sum_{k=0}^{n} B_k^n(a,b)g(k)$$

we can establish the same principle as (3.3) i.e.:

$$g(n) = 1, \quad n \geq 0 \text{ implies } \quad f(n) = \delta_0^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases} \tag{4.1}$$

provided that

$$f(n)\prod_{j=1}^{n}\binom{a+bj+j}{j} \neq 0. \tag{4.2}$$

Then in analogy to (3.4) we can show easily that

$$\sum_{k=0}^{n} B_k^n(a,b) = \delta_0^n. \tag{4.3}$$

Recurrence relations for effective computation of the Bs may be found in the same way that the special cases (3.5) and (3.6) were found. These are:

$$\sum_{k=j}^{n} \binom{a+bj+k}{j} B_k^n(a,b) = \delta_j^n \prod_{i=1}^{n}\binom{a+bi+i}{i} \tag{4.4}$$

and

$$\sum_{k=j}^{n} \binom{a+bk+n}{k} B_j^k(a,b) \prod_{i=1}^{k}\binom{a+bi+i}{i}^{-1} = \delta_0^n. \tag{4.5}$$

Here is a short table of the general inverse (1.3):

$$f(0) = g(0),$$

$$\binom{a+b+1}{1} f(1) = g(1) - g(0),$$

$$\binom{a+b+1}{1}\binom{a+2b+2}{2} f(2) = \binom{a+b+1}{1} g(2) - \binom{a+b+2}{1} g(1) + g(0)$$

$$\binom{a+b+1}{1}\binom{a+2b+2}{2}\binom{a+3b+3}{3} f(3) = \binom{a+b+1}{1}\binom{a+2b+2}{2} g(3)$$

$$- \binom{a+b+1}{1}\binom{a+2b+3}{2} g(2) + \frac{a+2b+2}{2}(a+3)g(1) + \frac{a+2b+2}{2}(a+2b-1)g(0).$$

Letting $a = 0$ and $b = 1$ in these we recover the first few lines of the pervious table for the inverse transform (3.2), however A_k^n differs from $B_k^n(a,b)$ in that certain factors have been removed in this special case allowing a simpler algebraic form for the As.

Write $P_n(a,b) = \prod_{i=1}^{n}\binom{a+bi+i}{i}$. Then consider the table of values:

n	$P_n(0,1)$	$\frac{(2n)!}{n!}$	$\frac{n!\,P_n(0,1)}{(2n)!}$
0	1	1	1
1	2	2	1
2	12	12	1
3	240	120	2
4	16800	1680	10
5	4233600	30240	140
6	3911846400	665280	5880
7	671272842240	17297280	38808

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Array of the coefficients A_k^n for $0 \le k \le n \le 10$

n	k = 0	1	2	3	4	5	6	7	8	9	10
0	1										
1	-1	1									
2	1	-3	2								
3	1	3	-10	6							
4	-1	9	10	-42	24						
5	-17	21	50	42	-216	120					
6	-107	-33	230	294	216	-1320	720				
7	-415	-1173	670	1974	1944	1320	-9360	5040			
8	1231	-13515	-4510	11130	17064	14520	9360	-75600	40320		
9	56671	-113739	-131230	20202	136296	157080	121680	75600	-685440	362880	
10	924365	-532209	-1976570	-901698	768312	1601160	1563120	1134000	685440	-6894720	3628800

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The numbers in the right-hand column are given by

$$\frac{n!\, P_n(0,1)}{(2n)!} = \frac{1}{n!} \prod_{i=1}^{n-1} \binom{2i}{i} \tag{4.6}$$

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