ON THE CONCRETE SECURITY OF LWE WITH SMALL SECRET

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Abstract. Lattice-based cryptography is currently under consideration for standardization in the ongoing NIST PQC Post-Quantum Cryptography competition, and is used as the basis for Homomorphic Encryption schemes world-wide. Both applications rely specifically on the hardness of the Learning With Errors (LWE) problem. Most Homomorphic Encryption deployments use small secrets as an optimization, so it is important to understand the concrete security of LWE when sampling the secret from a non-uniform, small distribution. Although there are numerous heuristics used to estimate the running time and quality of lattice reduction algorithms such as BKZ2.0, more work is needed to validate and test these heuristics in practice to provide concrete security parameter recommendations, especially in the case of small secret. In this work, we introduce a new approach which uses concrete attacks on the LWE problem as a way to study the performance and quality of BKZ2.0 directly. We find that the security levels for certain values of the modulus q and dimension n are smaller than predicted by the online LWE Estimator, due to the fact that the attacks succeed on these uSVP lattices for blocksizes which are smaller than expected based on current estimates. We also find that many instances of the TU Darmstadt LWE challenges can be solved significantly faster when the secret is chosen from the binary or ternary distributions.

1. Introduction

Lattice-based cryptography, proposed more than 20 years ago, is currently used as the basis for Homomorphic Encryption schemes world-wide. Cryptosystems based on the hardness of lattice problems are also under consideration for standardization in the ongoing NIST PQC Post-Quantum Cryptography competition. Both applications rely specifically on the hardness of the *Learning with Errors (LWE)* problem [Reg09].

Homomorphic Encryption allows computations on encrypted data, with security parameters for practical applications specified in HES, the Homomorphic Encryption Standard [ACC+18]. For efficiency reasons, it is common in homomorphic encryption to sample the secret from special distributions, such that it has small entries [BV11]. For example, two common distributions are the *binary* or *ternary* distributions [BLP+13, MP13], where the entries in the secret are in $\{0, 1\}$ or $\{0, \pm 1\}$ respectively. We also consider secrets sampled from the same small discrete gaussian distribution as the errors. In fact, the Homomorphic Encryption

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Standard [ACC+18] specifies tables of secure parameters for three possible distributions for the secret vector: uniform, ternary, and error distributions.

When the secret has a small norm, instances of LWE can be embedded into instances of the *unique Shortest Vector Problem (uSVP)* [BG14, AGVW17, BMW19]. To recover the shortest vector, lattice reduction algorithms such as the BKZ2.0 algorithm [CN11] are currently the most effective in practice. Although there are numerous heuristics used to estimate the running time and quality of lattice reduction algorithms such as BKZ2.0, [GN08, APS15, ADPS16], more work is needed to validate and test these heuristics in practice to provide concrete security parameter recommendations, especially in the case of small secret and small error.

In this work, we introduce a new approach which uses concrete attacks on the LWE problem as a way to study the performance and quality of BKZ2.0 directly. We generate random LWE instances using secrets sampled from binary, ternary or discrete Gaussian distributions. We convert each LWE instance into a uSVP instance and run the BKZ2.0 algorithm to find an approximation to the shortest vector. When the attack is successful, we can deduce a bound on the Hermite factor achieved for the given blocksize. In practice we find that the attacks succeed for a smaller block size than would be expected based on current estimates.

Our approach is similar to the approach taken in earlier work [LL15] for estimating the approximation factor for the LLL algorithm. Laine and Lauter used synthetically generated LWE instances to study the approximation factor for LLL in dimension up to 800, without solving the Shortest Vector Problem. They found that the approximation factor for LLL is significantly better than expected in dimensions up to 800, which confirmed and extended what Gama and Nyugen [GN08] had found for LLL in dimension up to 200. But it was not clear how that would extend to other lattice reduction algorithms such as BKZ. The attacks presented in [LL15] also cover the case of secrets sampled from the uniform distribution, but in that case the attacks are only successful for very large moduli.

In this work, we find that the security levels for certain values of the modulus q and dimension n are smaller than predicted by the online LWE Estimator [APS15]. This is due to the fact that the attacks succeed on these uSVP lattices for smaller blocksizes 30, 35, 40 and 45 than expected, for randomly generated LWE instances with small secret. The work of [BG14] attempts to quantify the loss of security when using binary secret by analyzing how much larger the lattice dimension n should be in order to achieve the same level of security. We use the same approach as [BG14] for attacking the LWE instances, but we run experiments to find the smallest blocksize necessary to break each LWE instance.

The tables of experimental data we present in Section 4 can be interpreted as follows: for each fixed blocksize β and lattice dimension n, the bold line in the table represents the smallest value of $\log(q)$ for which the attacks succeed. There are several estimates in the literature predicting which blocksize will be necessary to achieve a sufficiently good approximation factor for the attack to succeed (the 2008 [GN08] and the 2016 [ADPS16] estimates). However our experiments on LWE instances with small secret (and small error) show that the approximation factor may be significantly better than predicted by the estimates for random lattices, and this translates into attacks succeeding with smaller blocksize than expected.

For example, in Table 1 for binary secret, observe that blocksize 30 is enough to break LWE instances with $n = 120$ and $\log(q) = 12$ and error width $\sigma = 3.2$ in under

2 hours. Although machines are more powerful now, this can be compared with [BG14, Table 4] where the predicted security levels for $(n, q, \sigma) = (128, 2^{12}, 22.6)$, depending on the Hermite factor δ, range from $94 - 175$ bits of security for the standard attack to $34 - 59$ bits of security for their attack. Note that their $\delta \approx 1.008$ is closer to the delta we get for failed instances $\delta \approx 1.01$ than our average δ for successful cases $\delta \approx 0.99$.

We also observe a marked difference in blocksize required for a successful attack in comparison with the experiments presented in [AGVW17]. For example, in [AGVW17, Table 1], they validate the 2016 estimate in the case of $n = 110$, $\log(q) = 11$, where their attack requires blocksize 78. In our experiments attacking LWE instances with binary secrets, we successfully attack the same parameters with the same error width using blocksize 35 (see Table 2) with the dimension as predicted in the 2008 estimate. In this case the discrepancy is most likely due to the secret distribution: binary instead of uniform.

Our approach differs from the online LWE Estimator [APS15] in the sense that we run BKZ2.0 on synthetically generated LWE instances in order to study the approximation factor and the required blocksize, whereas the Estimator uses models based on heuristic estimates to predict the blocksize and running time necessary. We find for example that LWE instances in dimension 200 with $\log(q) = 19$ and binary secret can be broken using BKZ2.0 with blocksize 30, whereas the LWE Estimator predicts that blocksize 40 would be required, and a similar discrepancy with the LWE Estimator predictions applies to most entries in our Tables.

We present separate tables for each possible choice of the secret distribution: binary, ternary, and Gaussian, for blocksizes 30, 35, 40 and 45, and lattice dimension ranging from $n = 40$ to $n = 200$. Note the difference in security levels between the tables for binary, ternary, and Gaussian secrets. For the same choice of blocksize β and lattice dimension n, the attack succeeds for smaller values of $\log(q)$ for binary secret than for ternary secret and Gaussian secret (e.g. for $\beta = 30$, $n = 120$, $\log(q) = 12, 13, 14$ respectively).

We also generated synthetic instances of the TU Darmstadt LWE challenges [BBG$^+$16] with binary, ternary and discrete gaussian secrets, and ran our same attack on these instances. Although our experiments only cover blocksizes 30, 35, 40 and 45, these blocksizes are already large enough to attack all the solved LWE challenges in the online tables, for secrets sampled from the binary and ternary secret distributions. We observed significantly lower running times for successful attacks on instances generated with the binary distribution for the secret vector. We observed that sampling the secret from the discrete Gaussian error distribution yielded greater security than the binary or ternary distributions for the same set of parameters, as the attack rarely succeeds. Our attacks run in a matter of minutes (under an hour) for blocksizes 30, 35, 40 and in a matter of hours for blocksize 45, for the range of parameters where the actual challenges have been solved.

2. Preliminaries

Let $\mathbf{b}_1, \ldots, \mathbf{b}_d \in \mathbb{R}^d$ be linearly independent vectors, and let $\mathbf{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_d) \in \mathbb{R}^{d \times d}$ be the matrix whose columns are formed by them. The lattice generated by $\mathbf{B}$ is

$$L(\mathbf{B}) = \left\{ \mathbf{Bx} : \mathbf{x} \in \mathbb{Z}^d \right\} . \tag{2.1}$$

The *Shortest Vector Problem (SVP)* asks to find the shortest nonzero vector in the lattice, whose norm is the *first minimum*:

$$\lambda_1(L(\mathbf{B})) = \min_{\mathbf{v}\in L(\mathbf{B}),\mathbf{v}\neq 0} ||\mathbf{v}||\,, \tag{2.2}$$

where we use $||\cdot||$ to denote the ℓ_2-norm. Similarly, the *second minimum* is

$$\lambda_2(L(\mathbf{B})) = \min_{\mathbf{v}_1,\mathbf{v}_2\in L(\mathbf{B})} \left\{ \max\{||\mathbf{v}_1||, ||\mathbf{v}_2||\} \,:\, \mathbf{v}_1,\mathbf{v}_2 \text{ linearly independent} \right\}. \tag{2.3}$$

The *unique Shortest Vector Problem (uSVP)* with gap γ is a variant of the SVP where $\lambda_2 \geq \gamma \cdot \lambda_1$, for some $\gamma \geq 1$. While random lattices do not satisfy this condition, in Section 3 we describe a procedure for embedding an instance of LWE with small secrets to an instance of uSVP.

In this work, we use the BKZ2.0 lattice reduction algorithm [CN11] to solve instances of the uSVP. Let $\mathbf{b}_1^*,\ldots,\mathbf{b}_d^*$ denote the Gram-Schmidt orthogonalization of the basis vectors. For $1\leq i\leq d$, let π_i be the orthogonal projection over $(\mathbf{b}_1,\ldots,\mathbf{b}_{i-1})^\perp$. For $1\leq j\leq k\leq d$, let $B_{[j,k]}$ be the local projected block $(\pi_j(\mathbf{b}_j),\ldots,\pi_j(\mathbf{b}_k))$, and let $L_{[j,k]}$ be the lattice spanned by $B_{[j,k]}$, of dimension $k-j+1$.

Definition 2.1. A basis $\mathbf{b}_1,\ldots,\mathbf{b}_d$ is *BKZ-reduced* with blocksize $\beta\geq 2$ if it is LLL-reduced, and for each $1\leq j\leq d$, $||\mathbf{b}_j^*|| = \lambda_1(L_{[j,k]})$ where $k=\min(j+\beta-1,d)$.

The BKZ algorithm works by iteratively reducing each local block $B_{[j,k]}$ of size up to β. Each block is first LLL-reduced, before being enumerated to find a vector that is the shortest in the projected lattice $L_{[j,k]}$. The BKZ2.0 algorithm [CN11] improves on BKZ by modifying the enumeration routine, incorporating the sound pruning technique by [GNR10].

The volume of a lattice is $\mathrm{Vol}(L(\mathbf{B})) = |\det(\mathbf{B})|$. We use the root Hermite factor to measure the quality of the BKZ-reduced basis.

Definition 2.2. The *root Hermite factor* δ of a basis $\{\mathbf{b}_1,\ldots,\mathbf{b}_d\}$ is defined by

$$||\mathbf{b}_1|| = \delta^d\cdot \mathrm{Vol}(L(\mathbf{B}))^{1/d}\,. \tag{2.4}$$

For BKZ with block size β, Chen [Che13] gives the following estimate for δ which only depends on β.

$$\delta(\beta) \approx \left(\frac{\beta}{2\pi e}(\pi\beta)^{1/\beta}\right)^{\frac{1}{2(\beta-1)}}\,. \tag{2.5}$$

For a large β, we can approximate this by $\beta^{1/2\beta}$.

3. Reduction from LWE to uSVP

In this work, we study the uSVP attack on LWE, which is currently the most effective attack if the LWE secret has small entries [BG14, AGVW17, BMW19]. There are two known estimates for the conditions under which uSVP can be solved by lattice reduction, which are known as the *2008 estimate* [GN08] and the *2016 estimate* [ADPS16]. In this section, we describe the reduction from LWE to uSVP, which proceeds by first reducing LWE to BDD and then reducing BDD to uSVP. We also describe the 2008 and 2016 estimates, and calculate the optimal parameters for the uSVP attack under these estimates, as well as the predicted values of the Hermite factor.

3.1. **The LWE Problem.** We first define the search variant of the LWE problem.

Definition 3.1. Let $n \geq 1$, $q \geq 2$ be a prime modulus and let D_σ be a discrete gaussian distribution over $\mathbb{Z}$ with standard deviation σ. Let $A \in \mathbb{Z}_q^{m \times n}$ be a matrix with entries uniformly sampled from $\mathbb{Z}_q$, let $\mathbf{s} \in \mathbb{Z}_q^n$ be a secret vector, and let $\mathbf{e} \in \mathbb{Z}_q^m$ be an error vector with entries sampled independently from D_σ. Let $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \pmod q$. The goal of the LWE problem is to find $\mathbf{s}$, given $(\mathbf{A}, \mathbf{b})$.

We consider the following distributions for the secret:

- *Binary*: Secret has entries sampled uniformly at random from $\{0, 1\}$.
- *Ternary*: Secret has entries sampled uniformly at random from $\{0, \pm 1\}$.
- *Gaussian*: Secret has entries sampled from the same discrete gaussian distribution as the error.

3.2. **Reduction from LWE to BDD.** Assuming that the secret has a small norm, we can transform the LWE problem into the *Bounded Distance Decoding (BDD)* problem. Specifically, given a lattice $L(\mathbf{B})$ and a target vector $\mathbf{t}$, such that the distance of $\mathbf{t}$ from $L(\mathbf{B})$ is bounded by a factor of λ_1, the BDD problem asks to find a lattice vector $\mathbf{v} \in L(\mathbf{B})$ close to $\mathbf{t}$. Consider the lattice generated by

$$\mathbf{B}_0 = \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{A} & q \cdot \mathbf{I}_m \end{pmatrix}. \tag{3.1}$$

Since $\mathbf{A}\mathbf{s} + \mathbf{e} = \mathbf{b} \pmod q$, we can write $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} + q \cdot \mathbf{c}$ for some $\mathbf{c} \in \mathbb{Z}^m$. Hence the lattice contains the vector $\mathbf{B}_0 \begin{pmatrix} \mathbf{s} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{s} \\ \mathbf{A}\mathbf{s} + q\mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{s} \\ \mathbf{b} - \mathbf{e} \end{pmatrix}$. Thus if we solve the BDD problem in the lattice generated by $\mathbf{B}_0$, with respect to the target point $\mathbf{t} = \begin{pmatrix} \mathbf{0} \\ \mathbf{b} \end{pmatrix}$, then we obtain $\begin{pmatrix} \mathbf{s} \\ -\mathbf{e} \end{pmatrix}$, allowing us to recover the secret.

3.3. **Reduction from BDD to uSVP.** We can reduce the BDD problem to an instance of uSVP using Kannan's embedding technique [Kan87]. Consider the basis matrix obtained by adding one row and column to (3.1):

$$\mathbf{B}_1 = \begin{pmatrix} \mathbf{B}_0 & \mathbf{t} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & q \cdot \mathbf{I}_m & \mathbf{b} \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

The lattice generated by the columns of $\mathbf{B}_1$ contains the unique shortest vector

$$\mathbf{B}_1 \begin{pmatrix} \mathbf{s} \\ \mathbf{c} \\ -1 \end{pmatrix} = \begin{pmatrix} \mathbf{B}_0 \begin{pmatrix} \mathbf{s} \\ \mathbf{c} \end{pmatrix} - \mathbf{t} \\ -1 \end{pmatrix} = \begin{pmatrix} \mathbf{s} \\ -\mathbf{e} \\ -1 \end{pmatrix}. \tag{3.3}$$

Assuming that the gap between λ_1 and λ_2 in this lattice is sufficiently large, we can solve for the unique shortest vector using lattice reduction algorithms such as BKZ2.0. Following [BG14], we further optimize this by balancing the lengths of the secret and error vectors, scaling the secret by some constant factor ω. If the secret is sampled from the same discrete gaussian distribution as the error, then we set $\omega = 1$. For the binary or ternary secret distributions, consider the matrix

$$\mathbf{B} = \begin{pmatrix} \omega \cdot \mathbf{I}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & q \cdot \mathbf{I}_m & \mathbf{b} \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.4}$$

The lattice $L(\mathbf{B})$ generated by (3.4) has dimension

$$d = n + m + 1 \tag{3.5}$$

and contains a short vector

$$\mathbf{B}\begin{pmatrix}\mathbf{s}\\ \mathbf{c}\\ -1\end{pmatrix} = \begin{pmatrix}\omega\cdot\mathbf{s}\\ \mathbf{A}\mathbf{s}+q\mathbf{c}-\mathbf{b}\\ -1\end{pmatrix} = \begin{pmatrix}\omega\cdot\mathbf{s}\\ -\mathbf{e}\\ -1\end{pmatrix}. \tag{3.6}$$

Since this is the shortest vector of this lattice, we approximate the first minimum of the lattice by its expected norm:

$$\lambda_1 = \sqrt{\omega^2\cdot||\mathbf{s}||^2 + ||\mathbf{e}||^2 + 1} \approx \sqrt{\omega^2\cdot h + m\sigma^2 + 1}\,, \tag{3.7}$$

where σ is the standard deviation of the discrete Gaussian distribution and h is the expected value of $||\mathbf{s}||^2$. We have $h = \frac{n}{2}$ for the binary distribution and $h = \frac{2}{3}n$ for the ternary distribution.

We estimate the second minimum λ_2 to be the same as the first minimum of a random lattice with the same dimension using the *Gaussian Heuristic*. Since the lattice is q-ary, it also contains vectors of norm q, so we have

$$\lambda_2 \approx \min\left\{q, \sqrt{\frac{d}{2\pi e}}\omega^{n/d}q^{m/d}\right\}. \tag{3.8}$$

We can solve the uSVP using lattice reduction algorithms if λ_2 is sufficiently larger than λ_1. We choose ω to maximize the ratio $\frac{\lambda_2}{\lambda_1}$ as follows. First we write

$$\gamma = \frac{\lambda_2}{\lambda_1} \approx \frac{\min\left\{q, \sqrt{\frac{d}{2\pi e}}\omega^{n/d}q^{m/d}\right\}}{\sqrt{\omega^2 h + m\sigma^2}}. \tag{3.9}$$

We choose the parameters to optimize the second term in the minimum, since the Gaussian Heuristic would asymptotically be smaller than q. Differentiating the expression in (3.9) with respect to ω and setting the result to zero, we get

$$\omega^2 = \frac{nm}{h(d-n)}\sigma^2 \approx \frac{n}{h}\sigma^2\,. \tag{3.10}$$

This gives us $\omega = \sqrt{2}\sigma$ for the binary distribution and $\omega = \sqrt{\frac{3}{2}}\sigma$ for the ternary distribution. Substituting (3.10) into (3.7), we get

$$\lambda_1 \approx \sqrt{d}\sigma\,. \tag{3.11}$$

This also holds for the case where the secret is sampled from the same discrete gaussian distribution as the error. Notably, the shortest vector has the same ℓ_2-norm regardless of the secret distribution, whereas the ℓ_1-norm differs. Thus

$$\gamma = \frac{\min\left\{q, \sqrt{\frac{d}{2\pi e}}\omega^{n/d}q^{m/d}\right\}}{\sqrt{d}\sigma}. \tag{3.12}$$

Remark 3.2. Another commonly used secret distribution is the *uniform* distribution on $\mathbb{Z}_q$, where the entries of the secret are sampled uniformly at random from $\{0, 1, \ldots, q-1\}$. Since the secret does not have a small norm, the uSVP attack would require a much larger q to succeed. To balance the norms of the secret and error vectors, we have to choose the scaling factor to be $\omega \approx \frac{\sqrt{3}}{q}\sigma$. However, the

Gaussian heuristic would then be greater than q, and so $\lambda_2 = q$ from (3.8). For the uSVP attack to be effective, λ_2 would have to be much greater than λ_1, which means that q would have to be much larger than for the other secret distributions.

There are two known ways for estimating the conditions under which uSVP can be solved using lattice reduction, which are called the *2008 estimate* and the *2016 estimate* in the literature. We study each of these in turn.

3.4. **2008 estimate.** From experiments by Gama and Nguyen [GN08], they claimed that the shortest vector can be recovered if

$$\gamma = \frac{\lambda_2}{\lambda_1} \geq \delta^d \,, \tag{3.13}$$

where δ is the root Hermite factor of the lattice reduction algorithm, up to a multiplicative constant. In what follows, we will compute the estimate of δ based on the heuristic in (3.13) for our setting. We will fix n and q, while choosing the lattice dimension d to maximize γ. First we write

$$\gamma \approx \frac{\sqrt{\frac{d}{2\pi e}}\omega^{n/d} q^{m/d}}{\sqrt{d}\sigma} = \frac{1}{\sqrt{2\pi e}} \frac{\omega^{n/d} q^{m/d}}{\sigma} \approx \frac{1}{\sqrt{2\pi e}} \left(\frac{q}{\omega}\right)^{-n/d} \left(\frac{q}{\sigma}\right) \geq \delta^d \,. \tag{3.14}$$

We choose d to maximize the ratio in (3.14), by setting

$$d = \sqrt{\frac{n \log\left(\frac{q}{\omega}\right)}{\log \delta}} \,. \tag{3.15}$$

We solve for the largest possible value of δ as a function of n, q, ω, σ. First, we assume equality in (3.14) and take logarithms on both sides:

$$\log\left(\frac{q}{\sqrt{2\pi e}\sigma}\right) - \frac{n}{d}\log\left(\frac{q}{\omega}\right) = d \log \delta \,. \tag{3.16}$$

Substituting (3.15) and rearranging, we get the *2008 estimate* for δ:

$$\log \delta_{2008} = \frac{\log^2\left(\frac{q}{\sqrt{2\pi e}\sigma}\right)}{4n \log\left(\frac{q}{\omega}\right)} \,. \tag{3.17}$$

We substitute (3.17) into (3.15) to obtain

$$d_{2008} = \frac{2n \log\left(\frac{q}{\omega}\right)}{\log\left(\frac{q}{\sqrt{2\pi e}\sigma}\right)} \,. \tag{3.18}$$

This is the lattice dimension that we use in our experiments to compute δ_{2008}. We observe that δ_{2008} increases with q. For fixed n, β, we experimentally find the smallest q such that the attack succeeds. Substituting the parameters into (3.17), we then obtain a heuristic estimate of δ_{2008}, which we compare with the actual value of δ from (2.4).

We remark that (3.17) only holds for large q, such that λ_2 is given by the Gaussian Heuristic. If $\lambda_2 = q$, then the same analysis as above gives

$$\log \delta_{2008} = \frac{1}{d}\log\left(\frac{q}{\sqrt{d}\sigma}\right) \,. \tag{3.19}$$

We also compare δ_{2008} with the actual value of δ that we expect from the experiments, using the definition in (2.4) and assuming that the shortest vector is successfully recovered, and that λ_2 is equal to the Gaussian Heuristic. We have

$$\delta_{2008}^d = \frac{\lambda_2}{\lambda_1} = \sqrt{\frac{d}{2\pi e}}\delta^{-d}\,. \tag{3.20}$$

This gives us the relation between the expected experimental δ and δ_{2008}.

$$\delta = \frac{1}{\delta_{2008}}\left(\frac{d}{2\pi e}\right)^{1/2d}\,. \tag{3.21}$$

Hence we expect δ to trend differently from δ_{2008}.

3.5. **2016 estimate.** The *2016 estimate* is given in the New Hope key exchange paper [ADPS16]. The authors consider the evolution of the Gram-Schmidt coefficients of the unique shortest vector in the BKZ tours, assuming that the Geometric Series Assumption [Sch03] holds. This says that the norms of the Gram-Schmidt vectors after lattice reduction satisfy

$$||\mathbf{b}_i^*|| \approx \delta^{d-2i+2} \cdot \mathrm{Vol}(L(\mathbf{B}))^{1/d}\,. \tag{3.22}$$

The reasoning in [ADPS16] is that, if the projection of the unique shortest vector onto the space spanned by the last β Gram-Schmidt vectors is shorter than $\mathbf{b}_{d-\beta+1}^*$, then the SVP oracle in BKZ would be able to find it when called on the last block of size β. The success condition is thus given by

$$\sqrt{\frac{\beta}{d}}\lambda_1 \le ||\mathbf{b}_{d-\beta+1}^*||\,. \tag{3.23}$$

Based on these heuristics, we compute the estimated value of δ in our setting. Substituting $\lambda_1 \approx \sqrt{d}\sigma$ and (3.22), we get

$$\sqrt{\beta}\sigma \le \delta^{2\beta-d} \cdot \mathrm{Vol}(L(\mathbf{B}))^{1/d} = \delta^{2\beta-d}\omega^{n/d}q^{m/d}\,. \tag{3.24}$$

If we choose d to optimize this ratio, we obtain (3.15) again. Substituting (3.15) into (3.24) and taking logarithms, we get a quadratic equation in $\sqrt{\log\delta}$:

$$2\beta\log\delta - 2\sqrt{n\log\left(\frac{q}{\omega}\right)\log\delta} + \log\left(\frac{q}{\sqrt{\beta}\sigma}\right) = 0\,. \tag{3.25}$$

We solve this equation to get the *2016 estimate* for δ:

$$\log\delta_{2016} = \frac{n\log\left(\frac{q}{\omega}\right)}{4\beta^2}\left(1-\sqrt{1-\frac{2\beta\log\left(\frac{q}{\sqrt{\beta}\sigma}\right)}{n\log\left(\frac{q}{\omega}\right)}}\right)^2\,, \tag{3.26}$$

If the value inside the squareroot is negative, then we take $\log\delta_{2016} = \frac{n\log\left(\frac{q}{\omega}\right)}{4\beta^2}$. We obtain the lattice dimension d_{2016} by substituting (3.26) into (3.15). For large n, (3.26) is asymptotically

$$\log\delta_{2016} \approx \frac{\log^2\left(\frac{q}{\sqrt{\beta}\sigma}\right)}{4n\log\left(\frac{q}{\omega}\right)}\,. \tag{3.27}$$

We observe that (3.27) is similar to (3.17) except for the denominator of q in the numerator. The experiments in [AGVW17, BMW19] suggest that the 2016 estimate is more consistent with experiments than the 2008 estimate. In this paper, we will experimentally compare δ_{2008} and δ_{2016} with actual values of δ.

We compare δ_{2016} with the expected experimental value of δ, using the definition in (2.4) and assuming that the shortest vector is successfully recovered. We have

$$\delta_{2016}^{2\beta-d} = \sqrt{\frac{\beta}{d}} \frac{\lambda_1}{\mathrm{Vol}(L(\mathbf{B}))^{1/d}} = \sqrt{\frac{\beta}{d}} \delta^d \, . \tag{3.28}$$

Hence we have the relation

$$\delta = \delta_{2016}^{2\beta/d-1} \left(\frac{d}{\beta}\right)^{1/2d} . \tag{3.29}$$

We observe that δ trends differently from δ_{2016}, similarly to (3.21) for δ_{2008}.

4. Experiments

4.1. **Setup.** We perform our experiments using a 2.4 GHz Intel® Xeon® E5-2673 v4 processor, with 48 virtual CPUs and 192 GB of RAM. We generate random instances of LWE, and convert them into instances of uSVP via (3.4). We sample the errors from a discrete gaussian distribution with standard deviation $\sigma = 3.2$, using the discrete gaussian sampler in [The19], and we sample secrets uniformly from the binary, ternary and discrete gaussian distributions. To recover the shortest vector, we use the BKZ2.0 algorithm implemented in `fplll` [The16], with the `bkzautoabort` option, and with blocksizes $\beta = 30, 35, 40, 45$. The `bkzautoabort` option causes the algorithm to terminate when the norms of the Gram-Schmidt vectors stop changing.

For $\beta = 30$, we run experiments for n from 40 to 200 in steps of 10. For $\beta = 35, 40$, we choose n from 40 to 150, and for $\beta = 45$, we choose n from 40 to 100. We use a smaller range of values of n for higher β, since the running time of BKZ2.0 grows exponentially with β, so it is infeasible to run the experiments for higher β with large n. For each set of parameters, we vary $\log q$ to determine the smallest value of $\log q$ such that BKZ2.0 succeeds in recovering the secret. We perform 10 trials per set of parameters, to account for the randomness in sampling the lattices.

The data are in Tables 1 to 6, where the rows in boldface contain the data for the smallest value of $\log(q)$ where the attack succeeds. For each set of parameters, we compute the values of δ using the estimates in (3.17) and (3.26), which we tabulate as δ_{2008} and δ_{2016} respectively. Based on the estimates, we also compute the optimal values of the lattice dimensions from (3.15), which we tabulate as d_{2008} and d_{2016}. Since these dimensions are different, we conducted two sets of experiments for each set of parameters, where one set has lattice dimension d_{2008} and the other has dimension d_{2016}. We thus divide Tables 1 to 6 into two parts, where the left parts indicate the experiments for the 2008 estimate and the right for the 2016 estimate.

For each instance, we compute the actual values of δ using the definition in (2.4). We split the instances into cases where BKZ2.0 succeeds in recovering the secret, and cases where it fails, and we compute the average value of δ in each scenario. We tabulate these experimental values of δ under the columns labeled “Average successful δ” and “Average failed δ”.

Table 1. Binary secrets

β	n	$\log(q)$	d_{2008}	δ_{2008}	Number of successes	Average time (min)	Average successful δ	Average failed δ	d_{2016}	δ_{2016}	Number of successes	Average time (min)	Average successful δ	Average failed δ
		5	156	0.99952	0	1	-	1.00344	178	1.00255	0	2	-	1.00264
	40	6	**128**	**1.00484**	**5**	**1**	**1.00176**	**1.00661**	**114**	**1.00805**	**7**	**2**	**1.00277**	**1.00835**
		7	114	1.01014	10	1	0.99893	-	83	1.01933	10	1	1.00340	-
		6	160	1.00317	0	2	-	1.00525	157	1.00535	0	2	-	1.00545
	50	7	**143**	**1.00810**	**7**	**2**	**0.99992**	**1.00826**	**123**	**1.01096**	**8**	**2**	**1.00125**	**1.01118**
		8	133	1.01126	10	2	0.99722	-	105	1.01821	10	2	0.99952	-
		7	171	1.00671	0	3	-	1.00689	158	1.00793	0	3	-	1.00808
	60	8	**160**	**1.00937**	**8**	**4**	**0.99819**	**1.00950**	**138**	**1.01258**	**8**	**5**	**0.99903**	**1.01278**
		9	152	1.01219	10	4	0.99558	-	125	1.01808	10	4	0.99752	-
	70	8	186	1.00803	0	6	-	1.00816	169	1.00974	0	7	-	1.00990
		9	**177**	**1.01044**	**10**	**8**	**0.99672**	**-**	**155**	**1.01374**	**10**	**12**	**0.99780**	**-**
		8	213	1.00702	0	8	-	1.00709	200	1.00798	0	13	-	1.00805
	80	9	**203**	**1.00913**	**2**	**10**	**0.99701**	**1.00921**	184	1.01115	0	12	-	1.01123
		10	196	1.0112	10	12	0.99547	-	**173**	**1.01438**	**10**	**16**	**0.99607**	**-**
		9	228	1.00811	0	27	-	1.00820	212	1.00940	0	23	-	1.00949
	90	10	**221**	**1.00995**	**4**	**32**	**0.99628**	**1.01001**	**200**	**1.01207**	**6**	**27**	**0.99644**	**1.01224**
		11	215	1.01183	10	31	0.99452	-	192	1.01485	10	24	0.99525	-
		10	245	1.00895	0	45	-	1.00903	227	1.01041	0	42	-	1.01053
	100	11	**239**	**1.01064**	**7**	**55**	**0.99515**	**1.01073**	**218**	**1.01277**	**8**	**33**	**0.99579**	**1.01291**
		12	234	1.01235	10	54	0.99365	-	211	1.01520	10	35	0.99430	-
	110	11	263	1.00967	0	71	-	1.00973	244	1.01122	0	57	-	1.01131
		12	**258**	**1.01122**	**10**	**86**	**0.99442**	**-**	**237**	**1.01333**	**10**	**69**	**0.99504**	**-**
		11	287	1.00886	0	94	-	1.00890	270	1.01001	0	85	-	1.01006
30	120	12	**281**	**1.01028**	**2**	**106**	**0.99492**	**1.01035**	262	1.01187	0	75	-	1.01192
		13	277	1.01172	10	122	0.99372	-	**255**	**1.01378**	**10**	**138**	**0.99409**	**-**
		12	304	1.00949	0	78	-	1.00957	287	1.01071	0	112	-	1.01075
	130	13	**300**	**1.01081**	**2**	**141**	**0.99411**	**1.01085**	**280**	**1.01242**	**2**	**129**	**0.99452**	**1.01247**
		14	296	1.01214	10	174	0.99297	-	274	1.01413	10	148	0.99324	-
		13	323	1.01003	0	206	-	1.01007	304	1.01130	0	121	-	1.01138
	140	14	**319**	**1.01126**	**3**	**216**	**0.99360**	**1.01130**	**298**	**1.01286**	**8**	**220**	**0.99385**	**1.01296**
		15	315	1.01250	10	258	0.99257	-	294	1.01443	10	122	0.99301	-
		14	341	1.01051	0	281	-	1.01058	322	1.01180	0	174	-	1.01188
	150	15	**338**	**1.01166**	**8**	**315**	**0.99306**	**1.01170**	**317**	**1.01323**	**10**	**244**	**0.99333**	**-**
		16	335	1.01282	10	347	0.99196	-	313	1.01467	10	268	0.99224	-
	160	15	360	1.01093	0	253	-	1.01099	341	1.01222	0	334	-	1.01226
		16	**357**	**1.01201**	**10**	**397**	**0.99258**	**-**	**336**	**1.01355**	**10**	**368**	**0.99288**	**-**
	170	16	379	1.01130	0	531	-	1.01136	359	1.01259	0	546	-	1.01267
		17	**376**	**1.01232**	**10**	**516**	**0.99210**	**-**	**335**	**1.01382**	**10**	**422**	**0.99250**	**-**
		16	402	1.01067	0	609	-	1.01069	383	1.01175	0	484	-	1.01178
	180	17	**398**	**1.01163**	**2**	**626**	**0.99254**	**1.01170**	**378**	**1.01291**	**2**	**528**	**0.99268**	**1.01298**
		18	396	1.01260	10	739	0.99167	-	375	1.01406	10	392	0.99179	-
		17	421	1.01102	0	761	-	1.01103	401	1.01210	0	392	-	1.01217
	190	18	**418**	**1.01193**	**9**	**836**	**0.99217**	**1.01196**	**398**	**1.01319**	**10**	**851**	**0.99231**	**-**
		19	415	1.01285	10	937	0.99129	-	394	1.01427	10	881	0.99156	-
		18	440	1.01133	0	1183	-	1.01135	420	1.01241	0	951	-	1.01247
	200	19	**437**	**1.01220**	**6**	**1266**	**0.99169**	**1.01225**	**417**	**1.01343**	**10**	**1077**	**0.99200**	**-**
		20	435	1.01307	10	1450	0.99090	-	414	1.01446	10	1107	0.99114	-
		5	156	0.99952	0	1	-	1.00344	196	1.00210	0	2	-	1.00218
	40	6	**128**	**1.00484**	**8**	**2**	**1.00180**	**1.00661**	**114**	**1.00812**	**4**	**3**	**1.00234**	**1.00835**
		7	114	1.01014	10	2	0.99879	-	71	1.02702	10	0	1.00566	-
		6	160	1.00317	0	3	-	1.00525	161	1.00507	0	3	-	1.00518
	50	7	**143**	**1.00810**	**9**	**5**	**0.99989**	**1.00826**	**120**	**1.01159**	**10**	**2**	**1.00101**	**-**
		8	133	1.01126	10	4	0.99730	-	96	1.02173	10	2	0.99982	-
	60	7	171	1.00671	0	6	-	1.00689	158	1.00795	0	5	-	1.00808
		8	**160**	**1.00937**	**10**	**6**	**0.99829**	**-**	**134**	**1.01332**	**10**	**8**	**0.99996**	**-**
		7	200	1.00534	0	6	-	1.00585	194	1.00614	0	10	-	1.00622
35	70	8	186	1.00803	0	7	-	1.00816	**167**	**1.00994**	**2**	**11**	**0.99917**	**1.01014**
		9	**177**	**1.01044**	**10**	**9**	**0.99656**	**-**	151	1.01447	10	10	0.99747	-
		8	213	1.00702	0	12	-	1.00709	199	1.00799	0	15	-	1.00813
	80	9	**203**	**1.00913**	**5**	**13**	**0.99712**	**1.00921**	**181**	**1.01145**	**6**	**16**	**0.99777**	**1.01160**
		10	196	1.01120	10	24	0.99545	-	170	1.01504	10	15	0.99620	-
		9	228	1.00811	0	29	-	1.00820	211	1.00952	0	26	-	1.00958
	90	10	**221**	**1.00995**	**5**	**41**	**0.99620**	**1.01001**	**198**	**1.01240**	**10**	**33**	**0.99676**	**-**
		11	215	1.01183	10	43	0.99468	-	189	1.01544	10	30	0.99545	-
	100	10	245	1.00895	0	50	-	1.00903	225	1.01058	0	44	-	1.01072
		11	**239**	**1.01064**	**10**	**66**	**0.99526**	**-**	**215**	**1.01312**	**10**	**52**	**0.99580**	**-**

Table 2. Binary secrets (continued)

β	n	$\log(q)$	d_{2008}	δ_{2008}	Number of successes	Average time (min)	Average successful δ	Average failed δ	d_{2016}	δ_{2016}	Number of successes	Average time (min)	Average successful δ	Average failed δ
35	110	10	269	1.00814	0	65	-	1.00823	253	1.00924	0	83	-	1.00931
		11	**263**	**1.00967**	**2**	**51**	**0.99557**	**1.00973**	242	1.01142	0	66	-	1.01150
		12	258	1.01122	10	95	0.99449	-	**234**	**1.01367**	**10**	**59**	**0.99489**	**-**
	120	11	287	1.00886	0	111	-	1.00890	268	1.01012	0	88	-	1.01021
		12	**281**	**1.01028**	**2**	**113**	**0.99495**	**1.01035**	**259**	**1.01209**	**3**	**99**	**0.99544**	**1.01220**
		13	277	1.01172	10	137	0.99362	-	252	1.01411	10	91	0.99391	-
	130	12	304	1.00949	0	155	-	1.00957	285	1.01085	0	156	-	1.01090
		13	**300**	**1.01081**	**7**	**188**	**0.99430**	**1.01085**	**277**	**1.01264**	**7**	**194**	**0.99466**	**1.01274**
		14	296	1.01214	10	203	0.99309	-	271	1.01445	10	180	0.99358	-
	140	13	323	1.01003	0	217	-	1.01007	302	1.01146	0	182	-	1.01153
		14	**319**	**1.01126**	**8**	**265**	**0.99361**	**1.01130**	**296**	**1.01309**	**10**	**233**	**0.99396**	**-**
		15	315	1.01250	10	289	0.99250	-	291	1.01473	10	132	0.99281	-
	150	14	341	1.01051	0	312	-	1.01058	320	1.01196	0	243	-	1.01203
		15	**338**	**1.01166**	**10**	**350**	**0.99304**	**-**	**315**	**1.01345**	**10**	**305**	**0.99341**	**-**
40	40	5	156	0.99952	0	3	-	1.00344	216	1.00173	0	6	-	1.00179
		6	**128**	**1.00484**	**6**	**4**	**1.00202**	**1.00661**	**111**	**1.00857**	**6**	**6**	**1.00242**	**1.00881**
		7	114	1.01014	10	5	0.99919	-	80	1.02062	10	3	1.00158	-
	50	6	160	1.00317	0	6	-	1.00525	164	1.00488	0	9	-	1.00499
		7	**143**	**1.00810**	**10**	**11**	**0.99999**	**-**	**113**	**1.01297**	**10**	**12**	**1.00202**	**-**
	60	6	192	1.00217	0	9	-	1.00435	211	1.00353	0	14	-	1.00360
		7	**171**	**1.00671**	**1**	**11**	**1.00017**	**1.00689**	156	1.00811	0	15	-	1.00829
		8	160	1.00937	9	14	0.99821	1.00950	**128**	**1.01464**	**10**	**14**	**0.99965**	**-**
	70	7	200	1.00534	0	17	-	1.00585	195	1.00609	0	21	-	1.00616
		8	**186**	**1.00803**	**5**	**29**	**0.99886**	**1.00816**	**164**	**1.01032**	**4**	**23**	**0.99954**	**1.01051**
		9	177	1.01044	10	32	0.99666	-	145	1.01562	10	17	0.99817	-
	80	8	213	1.00702	0	38	-	1.00709	198	1.00810	0	32	-	1.00821
		9	**203**	**1.00913**	**9**	**46**	**0.99727**	**1.00921**	**178**	**1.01193**	**8**	**36**	**0.99828**	**1.01200**
		10	196	1.01120	10	52	0.99552	-	164	1.01601	10	35	0.99651	-
	90	9	228	1.00811	0	54	-	1.00820	208	1.00974	0	64	-	1.00986
		10	**221**	**1.00995**	**10**	**72**	**0.99628**	**-**	**194**	**1.01290**	**10**	**55**	**0.99703**	**-**
	100	9	253	1.00730	0	79	-	1.00738	238	1.00825	0	82	-	1.00835
		10	245	1.00895	0	81	-	1.00903	**223**	**1.01085**	**2**	**74**	**0.99722**	**1.01091**
		11	**239**	**1.01064**	**10**	**105**	**0.99524**	**-**	212	1.01360	10	74	0.99599	-
	110	10	269	1.00814	0	111	-	1.00823	251	1.00939	0	109	-	1.00946
		11	**263**	**1.00967**	**8**	**117**	**0.99574**	**1.00973**	**239**	**1.01172**	**3**	**119**	**0.99612**	**1.01179**
		12	258	1.01122	10	147	0.99442	-	230	1.01413	10	121	0.99520	-
	120	11	287	1.00886	0	185	-	1.00890	266	1.01031	0	125	-	1.01037
		12	**281**	**1.01028**	**8**	**196**	**0.99508**	**1.01035**	**256**	**1.01240**	**7**	**148**	**0.99515**	**1.01249**
		13	277	1.01172	10	235	0.99374	-	249	1.01454	10	145	0.99409	-
	130	12	304	1.00949	0	235	-	1.00957	282	1.01105	0	195	-	1.01113
		13	**300**	**1.01081**	**10**	**296**	**0.99428**	**-**	**274**	**1.01295**	**10**	**210**	**0.99467**	**-**
	140	12	328	1.00881	0	242	-	1.00885	308	1.00998	0	260	-	1.01004
		13	**323**	**1.01003**	**1**	**300**	**0.99479**	**1.01007**	299	1.01167	0	295	-	1.01176
		14	319	1.01126	10	372	0.99361	-	**292**	**1.01339**	**10**	**391**	**0.99393**	**-**
	150	13	346	1.00936	0	402	-	1.00940	325	1.01063	0	521	-	1.01066
		14	**341**	**1.01051**	**6**	**424**	**0.99412**	**1.01058**	**317**	**1.01218**	**5**	**348**	**0.99452**	**1.01226**
		15	338	1.01166	10	420	0.99328	-	311	1.01375	10	361	0.99364	-
45	40	5	156	0.99952	0	23	-	1.00344	238	1.00142	0	31	-	1.00148
		6	**128**	**1.00484**	**10**	**44**	**1.00187**	**-**	**101**	**1.01033**	**10**	**45**	**1.00366**	**-**
	50	6	160	1.00317	0	69	-	1.00525	166	1.00474	0	64	-	1.00487
		7	**143**	**1.00810**	**10**	**106**	**0.99988**	**-**	**94**	**1.01874**	**10**	**72**	**1.00434**	**-**
	60	6	192	1.00217	0	104	-	1.00435	217	1.00334	0	164	-	1.00340
		7	**171**	**1.00671**	**3**	**165**	**1.00038**	**1.00689**	153	1.00846	0	137	-	1.00862
		8	160	1.00937	10	166	0.99804	-	**118**	**1.01737**	**10**	**132**	**1.00113**	**-**
	70	7	200	1.00534	0	167	-	1.00585	194	1.00611	0	149	-	1.00622
		8	**186**	**1.00803**	**6**	**228**	**0.99869**	**1.00816**	**160**	**1.01094**	**10**	**222**	**0.99947**	**-**
		9	177	1.01044	10	264	0.99667	-	137	1.01755	10	160	0.99959	-
	80	8	213	1.00702	0	284	-	1.00709	196	1.00831	0	320	-	1.00838
		9	**203**	**1.00913**	**10**	**340**	**0.99741**	**-**	**172**	**1.01265**	**10**	**283**	**0.99840**	**-**
	90	9	228	1.00811	0	418	-	1.00820	205	1.01007	0	384	-	1.01015
		10	**221**	**1.00995**	**10**	**450**	**0.99616**	**-**	**189**	**1.01360**	**10**	**401**	**0.99718**	**-**
	100	9	253	1.00730	0	411	-	1.00738	236	1.00841	0	512	-	1.00849
		10	**245**	**1.00895**	**4**	**562**	**0.99665**	**1.00903**	219	1.01123	0	496	-	1.01132
		11	239	1.01064	10	659	0.99520	-	**207**	**1.01426**	**10**	**557**	**0.99624**	**-**

Table 3. Ternary secrets

β	n	$\log(q)$	d_{2008}	δ_{2008}	Number of successes	Average time (min)	Average successful δ	Average failed δ	d_{2016}	δ_{2016}	Number of successes	Average time (min)	Average successful δ	Average failed δ
30	40	5	173	0.99927	0	1	-	1.00348	203	1.00218	0	2	-	1.00253
		6	**139**	**1.00416**	**2**	**1**	**1.00148**	**1.00667**	**129**	**1.00683**	**5**	**2**	**1.00271**	**1.00775**
		7	122	1.00949	10	2	0.99893	-	95	1.01570	10	1	1.00126	-
	50	6	174	1.00267	0	2	-	1.00528	174	1.00469	0	3	-	1.00528
		7	**152**	**1.00758**	**7**	**3**	**0.99991**	**1.00842**	**135**	**1.00967**	**3**	**2**	**1.00077**	**1.01069**
		8	141	1.01065	10	3	0.99729	-	115	1.01609	10	2	0.99900	-
	60	7	183	1.00608	0	4	-	1.00694	172	1.00714	0	5	-	1.00785
		8	**169**	**1.00887**	**10**	**6**	**0.99812**	**-**	**149**	**1.01143**	**7**	**6**	**0.99868**	**1.01235**
		9	159	1.01163	10	6	0.99574	-	134	1.01655	10	7	0.99634	-
	70	8	197	1.00759	0	7	-	1.00820	181	1.00895	0	9	-	1.00973
		9	**186**	**1.00996**	**10**	**10**	**0.99684**	**-**	**165**	**1.01274**	**10**	**14**	**0.99735**	**-**
	80	9	212	1.00871	0	16	-	1.00936	195	1.01040	0	27	-	1.01108
		10	**204**	**1.01075**	**10**	**23**	**0.99573**	**-**	**182**	**1.01351**	**10**	**20**	**0.99659**	**-**
	90	9	239	1.00774	0	30	-	1.00827	224	1.00881	0	33	-	1.00942
		10	**230**	**1.00955**	**4**	**40**	**0.99631**	**1.01012**	211	1.01138	0	21	-	1.01203
		11	223	1.01141	10	44	0.99468	-	**201**	**1.01408**	**10**	**23**	**0.99554**	**-**
	100	10	255	1.00859	0	58	-	1.00913	239	1.00985	0	69	-	1.01040
		11	**248**	**1.01026**	**7**	**62**	**0.99530**	**1.01080**	**228**	**1.01215**	**6**	**57**	**0.99580**	**1.01279**
		12	242	1.01195	10	69	0.99380	-	220	1.01452	10	50	0.99439	-
	110	11	273	1.00932	0	84	-	1.00979	255	1.01069	0	73	-	1.01122
		12	**266**	**1.01086**	**9**	**94**	**0.99462**	**1.01141**	**246**	**1.01276**	**10**	**92**	**0.99483**	**-**
		13	261	1.01241	10	100	0.99313	-	239	1.01487	10	93	0.99360	-
	120	12	290	1.00995	0	116	-	1.01046	272	1.01138	0	102	-	1.01189
		13	**285**	**1.01137**	**9**	**143**	**0.99382**	**1.01187**	**264**	**1.01325**	**10**	**110**	**0.99407**	**-**
		14	281	1.01280	10	159	0.99252	-	258	1.01514	10	137	0.99295	-
	130	13	309	1.01049	0	152	-	1.01093	289	1.01195	0	144	-	1.01250
		14	**304**	**1.01181**	**10**	**200**	**0.99319**	**-**	**283**	**1.01365**	**10**	**173**	**0.99366**	**-**
	140	13	332	1.00974	0	225	-	1.01019	314	1.01089	0	219	-	1.01139
		14	**327**	**1.01096**	**5**	**263**	**0.99376**	**1.01142**	308	1.01243	0	153	-	1.01289
		15	323	1.01219	10	286	0.99262	-	**302**	**1.01398**	**10**	**181**	**0.99296**	**-**
	150	14	351	1.01023	0	315	-	1.01061	332	1.01141	0	286	-	1.01187
		15	**346**	**1.01137**	**7**	**354**	**0.99314**	**1.01181**	**326**	**1.01283**	**10**	**354**	**0.99354**	**-**
		16	343	1.01252	10	386	0.99207	-	321	1.01426	10	288	0.99246	-
	160	15	369	1.01065	0	372	-	1.01106	350	1.01185	0	353	-	1.01231
		16	**365**	**1.01173**	**10**	**465**	**0.99269**	**-**	**345**	**1.01317**	**10**	**466**	**0.99296**	**-**
	170	16	388	1.01104	0	584	-	1.01142	369	1.01224	0	455	-	1.01264
		17	**385**	**1.01205**	**10**	**639**	**0.99228**	**-**	**364**	**1.01347**	**10**	**528**	**0.99259**	**-**
	180	16	411	1.01042	0	758	-	1.01077	393	1.01143	0	675	-	1.01179
		17	**407**	**1.01138**	**2**	**756**	**0.99272**	**1.01175**	387	1.01258	0	680	-	1.01300
		18	404	1.01234	10	855	0.99179	-	**383**	**1.01373**	**10**	**808**	**0.99218**	**-**
	190	17	430	1.01078	0	857	-	1.01110	411	1.01180	0	708	-	1.01216
		18	**426**	**1.01169**	**6**	**930**	**0.99225**	**1.01205**	406	1.01287	0	639	-	1.01328
		19	423	1.01260	10	986	0.99144	-	**402**	**1.01395**	**10**	**866**	**0.99174**	**-**
	200	18	449	1.01110	0	1320	-	1.01141	430	1.01212	0	1290	-	1.01245
		19	**446**	**1.01197**	**6**	**1498**	**0.99197**	**1.01228**	**425**	**1.01314**	**10**	**1156**	**0.99209**	**-**
		20	443	1.01284	10	1426	0.99106	-	422	1.01416	10	765	0.99116	-
35	40	5	173	0.99927	0	2	-	1.00348	224	1.00179	0	4	-	1.00208
		6	**139**	**1.00416**	**5**	**2**	**1.00181**	**1.00667**	**131**	**1.00668**	**5**	**3**	**1.00203**	**1.00751**
		7	122	1.00949	10	2	0.99901	-	85	1.01988	10	1	1.00321	-
	50	6	174	1.00267	0	3	-	1.00528	180	1.00441	0	4	-	1.00494
		7	**152**	**1.00758**	**5**	**4**	**1.00009**	**1.00842**	**133**	**1.00998**	**6**	**4**	**1.00054**	**1.01101**
		8	141	1.01065	10	4	0.99736	-	108	1.01816	10	2	0.99953	-
	60	7	183	1.00608	0	6	-	1.00694	173	1.00708	0	9	-	1.00776
		8	**169**	**1.00887**	**10**	**7**	**0.99831**	**-**	**146**	**1.01192**	**9**	**6**	**0.99935**	**1.01267**
	70	8	197	1.00759	0	10	-	1.00820	180	1.00906	0	12	-	1.00983
		9	**186**	**1.00996**	**10**	**14**	**0.99688**	**-**	**161**	**1.01328**	**10**	**14**	**0.99752**	**-**
	80	8	225	1.00664	0	21	-	1.00717	214	1.00735	0	23	-	1.00793
		9	**212**	**1.00871**	**2**	**26**	**0.99750**	**1.00936**	**193**	**1.01062**	**1**	**15**	**0.99757**	**1.01131**
		10	204	1.01075	10	29	0.99569	-	179	1.01403	10	25	0.99637	-
	90	9	239	1.00774	0	40	-	1.00827	223	1.00888	0	24	-	1.00950
		10	**230**	**1.00955**	**7**	**53**	**0.99636**	**1.01165**	**208**	**1.01165**	**8**	**31**	**0.99691**	**1.01238**
		11	223	1.01141	10	54	0.99473	-	198	1.01458	10	27	0.99554	-
	100	10	255	1.00859	0	65	-	1.00913	237	1.00997	0	51	-	1.01058
		11	**248**	**1.01026**	**10**	**85**	**0.99537**	**-**	**225**	**1.01234**	**10**	**62**	**0.99591**	**-**

Table 4. Ternary secrets (continued)

β	n	$\log(q)$	d_{2008}	δ_{2008}	Number of successes	Average time (min)	Average successful δ	Average failed δ	d_{2016}	δ_{2016}	Number of successes	Average time (min)	Average successful δ	Average failed δ
35	110	11	273	1.00932	0	48	-	1.00979	253	1.01085	0	62	-	1.01140
		12	**266**	**1.01086**	**10**	**95**	**0.99459**	**-**	**243**	**1.01305**	**10**	**82**	**0.99468**	**-**
	120	11	297	1.00854	0	136	-	1.00901	280	1.00964	0	118	-	1.01014
		12	**290**	**1.00995**	**1**	**142**	**0.99525**	**1.01046**	**269**	**1.01156**	**3**	**120**	**0.99542**	**1.01216**
		13	285	1.01137	10	168	0.99382	-	261	1.01354	10	113	0.99422	-
	130	12	315	1.00918	0	172	-	1.00959	296	1.01039	0	147	-	1.01087
		13	**309**	**1.01049**	**4**	**203**	**0.99437**	**1.01093**	**287**	**1.01215**	**4**	**157**	**0.99483**	**1.01268**
		14	304	1.01181	10	251	0.99313	-	280	1.01393	10	188	0.99371	-
	140	13	332	1.00974	0	260	-	1.01019	313	1.01102	0	218	-	1.01147
		14	**327**	**1.01096**	**8**	**302**	**0.99383**	**1.01142**	**305**	**1.01263**	**10**	**180**	**0.99404**	**-**
		15	323	1.01219	10	306	0.99256	-	299	1.01425	10	233	0.99289	-
	150	14	351	1.01023	0	352	-	1.01061	330	1.01155	0	303	-	1.01202
		15	**346**	**1.01137**	**10**	**417**	**0.99324**	**-**	**324**	**1.01303**	**10**	**365**	**0.99350**	**-**
40	40	5	173	0.99927	0	4	-	1.00348	246	1.00147	0	12	-	1.00172
		6	**139**	**1.00416**	**8**	**6**	**1.00180**	**1.00667**	**131**	**1.00667**	**6**	**8**	**1.00248**	**1.00751**
		7	122	1.00949	10	6	0.99880	-	81	1.02205	10	2	1.00269	-
	50	6	174	1.00267	0	7	-	1.00528	184	1.00418	0	13	-	1.00472
		7	**152**	**1.00758**	**7**	**10**	**0.99982**	**1.00842**	**129**	**1.01063**	**10**	**11**	**1.00082**	**-**
		8	141	1.01065	10	11	0.99731	-	94	1.02384	10	6	1.00122	-
	60	6	209	1.00179	0	18	-	1.00437	234	1.00310	0	24	-	1.00349
		7	**183**	**1.00608**	**1**	**12**	**1.00002**	**1.00694**	172	1.00713	0	16	-	1.00785
		8	169	1.00887	10	13	0.99811	-	**141**	**1.01275**	**10**	**15**	**0.99952**	**-**
	70	7	213	1.00487	0	28	-	1.00595	212	1.00546	0	34	-	1.00601
		8	**197**	**1.00759**	**1**	**33**	**0.99829**	**1.00820**	**178**	**1.00930**	**2**	**27**	**0.99904**	**1.01006**
		9	186	1.00996	10	35	0.99666	-	156	1.01412	10	23	0.99772	-
	80	8	225	1.00664	0	43	-	1.00717	213	1.00740	0	43	-	1.00800
		9	**212**	**1.00871**	**5**	**52**	**0.99735**	**1.00936**	**189**	**1.01097**	**2**	**42**	**0.99838**	**1.01179**
		10	204	1.01075	10	56	0.99566	-	174	1.01480	10	50	0.99665	-
	90	9	239	1.00774	0	63	-	1.00827	221	1.00903	0	60	-	1.00968
		10	**230**	**1.00955**	**8**	**86**	**0.99622**	**1.01012**	**205**	**1.01204**	**7**	**66**	**0.99673**	**1.01237**
		11	223	1.01141	10	93	0.99468	-	193	1.01527	10	76	0.99537	-
	100	9	265	1.00696	0	95	-	1.00746	253	1.00770	0	84	-	1.00819
		10	**255**	**1.00859**	**1**	**106**	**0.99648**	**1.00913**	235	1.01019	0	105	-	1.01076
		11	248	1.01026	10	124	0.99540	-	**222**	**1.01284**	**10**	**132**	**0.99573**	**-**
	110	11	273	1.00932	0	134	-	1.00979	250	1.01110	0	134	-	1.01168
		12	**266**	**1.01086**	**10**	**171**	**0.99461**	**-**	**239**	**1.01344**	**10**	**140**	**0.99529**	**-**
	120	11	297	1.00854	0	170	-	1.00901	278	1.00979	0	189	-	1.01029
		12	**290**	**1.00995**	**4**	**218**	**0.99502**	**1.01046**	**267**	**1.01185**	**4**	**207**	**0.99541**	**1.01235**
		13	285	1.01137	10	233	0.99385	-	258	1.01392	10	185	0.99443	-
	130	12	315	1.00918	0	288	-	1.00959	293	1.01056	0	166	-	1.01109
		13	**309**	**1.01049**	**8**	**304**	**0.99437**	**1.01093**	**284**	**1.01241**	**10**	**205**	**0.99474**	**-**
		14	304	1.01181	10	356	0.99311	-	277	1.01430	10	236	0.99377	-
	140	13	332	1.00974	0	369	-	1.01019	310	1.01121	0	376	-	1.01169
		14	**327**	**1.01096**	**10**	**436**	**0.99380**	**-**	**302**	**1.01289**	**10**	**412**	**0.99399**	**-**
	150	13	356	1.00909	0	479	-	1.00948	336	1.01022	0	363	-	1.01065
		14	**351**	**1.01023**	**1**	**470**	**0.99412**	**1.01061**	327	1.01175	0	343	-	1.01224
		15	346	1.01137	10	500	0.99323	-	**320**	**1.01329**	**10**	**367**	**0.99379**	**-**
45	40	5	173	0.99927	0	44	-	1.00348	272	1.00121	0	54	-	1.00141
		6	**139**	**1.00416**	**7**	**53**	**1.00188**	**1.00667**	**129**	**1.00687**	**10**	**57**	**1.00261**	**-**
		7	122	1.00949	10	95	0.99884	-	90	1.01738	10	73	0.99989	-
	50	6	174	1.00267	0	89	-	1.00528	188	1.00400	0	105	-	1.00452
		7	**152**	**1.00758**	**10**	**135**	**0.99984**	**-**	**121**	**1.01200**	**10**	**137**	**1.00121**	**-**
	60	7	183	1.00608	0	180	-	1.00694	170	1.00728	0	139	-	1.00804
		8	**169**	**1.00887**	**10**	**201**	**0.99827**	**-**	**133**	**1.01422**	**10**	**184**	**1.00026**	**-**
	70	7	213	1.00487	0	217	-	1.00595	213	1.00542	0	241	-	1.00595
		8	**197**	**1.00759**	**3**	**270**	**0.99869**	**1.00820**	**174**	**1.00970**	**7**	**306**	**0.99933**	**1.01053**
		9	186	1.00996	10	329	0.99674	-	150	1.01544	10	263	0.99816	-
	80	8	225	1.00664	0	344	-	1.00717	211	1.00752	0	285	-	1.00815
		9	**212**	**1.00871**	**9**	**404**	**0.99749**	**1.00936**	**185**	**1.01150**	**10**	**381**	**0.99828**	**-**
		10	204	1.01075	10	444	0.99575	-	168	1.01590	10	328	0.99736	-
	90	9	239	1.00774	0	430	-	1.00827	218	1.00927	0	395	-	1.00995
		10	**230**	**1.00955**	**10**	**542**	**0.99641**	**-**	**200**	**1.01259**	**10**	**416**	**0.99740**	**-**
	100	10	255	1.00859	0	583	-	1.00913	231	1.01049	0	570	-	1.01114
		11	**248**	**1.01026**	**10**	**739**	**0.99528**	**-**	**217**	**1.01337**	**10**	**714**	**0.99624**	**-**

TABLE 5. Gaussian secrets

β	n	$\log(q)$	d_{2008}	δ_{2008}	Number of successes	Average time (min)	Average successful δ	Average failed δ	d_{2016}	δ_{2016}	Number of successes	Average time (min)	Average successful δ	Average failed δ
30	40	7	170	1.00677	0	3	-	1.00694	157	1.00799	0	4	-	1.00814
		8	**150**	**1.00998**	**10**	**3**	**0.99758**	**-**	**126**	**1.01413**	**10**	**4**	**0.99918**	**-**
	50	7	213	1.00487	0	5	-	1.00550	207	1.00571	0	11	-	1.00582
		8	**187**	**1.00798**	**4**	**7**	**0.99833**	**1.00813**	171	1.00965	0	7	-	1.00973
		9	171	1.01086	10	8	0.99621	-	**147**	**1.01467**	**10**	**6**	**0.99726**	**-**
	60	8	225	1.00664	0	10	-	1.00671	213	1.00738	0	16	-	1.00749
		9	**205**	**1.00904**	**5**	**11**	**0.99709**	**1.00912**	**186**	**1.01099**	**7**	**16**	**0.99800**	**1.01109**
		10	192	1.01148	10	10	0.99506	-	168	1.01492	10	12	0.99561	-
	70	9	239	1.00775	0	18	-	1.00781	224	1.00882	0	23	-	1.00889
		10	**223**	**1.00983**	**4**	**35**	**0.99609**	**1.00996**	**204**	**1.01185**	**5**	**25**	**0.99659**	**1.01191**
		11	212	1.01200	10	31	0.99415	-	189	1.01516	10	22	0.99487	-
	80	10	255	1.00859	0	46	-	1.00868	238	1.00985	0	32	-	1.00997
		11	**242**	**1.01049**	**9**	**52**	**0.99521**	**1.01060**	**222**	**1.01253**	**10**	**42**	**0.99544**	**-**
		12	232	1.01245	10	56	0.99341	-	209	1.01537	10	43	0.99389	-
	90	11	272	1.00932	0	69	-	1.00943	255	1.01069	0	64	-	1.01073
		12	**261**	**1.01106**	**10**	**87**	**0.99419**	**-**	**241**	**1.01307**	**10**	**75**	**0.99463**	**-**
	100	12	290	1.00995	0	107	-	1.01004	272	1.01138	0	133	-	1.01142
		13	**281**	**1.01155**	**10**	**126**	**0.99363**	**-**	**260**	**1.01352**	**10**	**90**	**0.99387**	**-**
	110	12	319	1.00904	0	138	-	1.00912	303	1.01008	0	116	-	1.01011
		13	**309**	**1.01049**	**1**	**156**	**0.99435**	**1.01053**	289	1.01195	0	136	-	1.01205
		14	300	1.01196	10	172	0.99298	-	**279**	**1.01388**	**10**	**144**	**0.99346**	**-**
	120	13	337	1.00961	0	209	-	1.00965	319	1.01072	0	228	-	1.01077
		14	**327**	**1.01096**	**3**	**235**	**0.99367**	**1.01104**	**308**	**1.01243**	**7**	**212**	**0.99396**	**1.01246**
		15	320	1.01232	10	253	0.99248	-	298	1.01417	10	211	0.99292	-
	130	14	355	1.01011	0	172	-	1.01014	336	1.01126	0	193	-	1.01133
		15	**346**	**1.01137**	**9**	**327**	**0.99318**	**1.01144**	**326**	**1.01283**	**7**	**292**	**0.99352**	**1.01290**
		16	339	1.01264	10	358	0.99195	-	318	1.01443	10	285	0.99215	-
	140	15	373	1.01055	0	423	-	1.01059	354	1.01172	0	390	-	1.01177
		16	**365**	**1.01173**	**9**	**487**	**0.99266**	**1.01181**	**345**	**1.01317**	**10**	**405**	**0.99279**	**-**
		17	359	1.01292	10	499	0.99151	-	337	1.01465	10	429	0.99184	-
	150	15	399	1.00985	0	524	-	1.00991	382	1.01078	0	369	-	1.01082
		16	**391**	**1.01095**	**1**	**516**	**0.99319**	**1.01101**	**372**	**1.01212**	**2**	**601**	**0.99333**	**1.01217**
		17	385	1.01205	10	560	0.99214	-	364	1.01347	10	430	0.99242	-
	160	17	410	1.01130	0	691	-	1.01135	391	1.01247	0	344	-	1.01249
		18	**404**	**1.01234**	**10**	**860**	**0.99177**	**-**	**383**	**1.01373**	**10**	**401**	**0.99195**	**-**
	170	17	436	1.01063	0	1067	-	1.01066	417	1.01160	0	838	-	1.01166
		18	**429**	**1.01161**	**1**	**949**	**0.99232**	**1.01166**	**409**	**1.01277**	**2**	**587**	**0.99220**	**1.01284**
		19	423	1.01260	10	1168	0.99134	-	402	1.01395	10	688	0.99158	-
	180	18	454	1.01096	0	1256	-	1.01102	435	1.01194	0	1285	-	1.01201
		19	**448**	**1.01190**	**2**	**1533**	**0.99184**	**1.01195**	**428**	**1.01305**	**7**	**1188**	**0.99209**	**1.01310**
		20	443	1.01284	10	1600	0.99102	-	422	1.01416	10	694	0.99103	-
	190	19	473	1.01127	0	1642	-	1.01131	454	1.01225	0	1445	-	1.01228
		20	**467**	**1.01216**	**1**	**1871**	**0.99153**	**-**	**447**	**1.01329**	**10**	**1023**	**0.99178**	**-**
		21	462	1.01305	10	1853	0.99057	-	441	1.01434	10	1698	0.99098	-
	200	20	492	1.01155	0	1969	-	1.01158	472	1.01252	0	1757	-	1.01259
		21	**487**	**1.01239**	**10**	**2249**	**0.99123**	**-**	**466**	**1.01351**	**10**	**1124**	**0.99131**	**-**
35	40	6	207	1.00183	0	4	-	1.00403	225	1.00335	0	13	-	1.00341
		7	**170**	**1.00677**	**1**	**4**	**0.99939**	**1.00694**	157	1.00802	0	9	-	1.00814
		8	150	1.00998	10	5	0.99750	-	**121**	**1.01534**	**10**	**3**	**0.99973**	**-**
	50	7	213	1.00487	0	7	-	1.00550	210	1.00556	0	12	-	1.00565
		8	**187**	**1.00798**	**2**	**8**	**0.99874**	**1.00813**	**169**	**1.00984**	**7**	**13**	**0.99870**	**1.00996**
		9	171	1.01086	10	8	0.99614	-	143	1.01560	10	8	0.99759	-
	60	8	225	1.00664	0	13	-	1.00671	214	1.00735	0	22	-	1.00742
		9	**205**	**1.00904**	**6**	**13**	**0.99701**	**1.00912**	**183**	**1.01127**	**6**	**19**	**0.99776**	**1.01146**
		10	192	1.01148	10	19	0.99512	-	164	1.01567	10	10	0.99641	-
	70	9	239	1.00775	0	34	-	1.00781	223	1.00889	0	26	-	1.00897
		10	**223**	**1.00983**	**8**	**33**	**0.99611**	**1.00996**	**201**	**1.01216**	**10**	**39**	**0.99682**	**-**
		11	212	1.0120	10	36	0.99425	-	185	1.01580	10	20	0.99516	-
	80	10	255	1.00859	0	57	-	1.00868	237	1.00998	0	49	-	1.01006
		11	**242**	**1.01049**	**10**	**69**	**0.99516**	**-**	**219**	**1.01285**	**10**	**50**	**0.99573**	**-**
	90	11	272	1.00932	0	84	-	1.00943	253	1.01085	0	76	-	1.01090
		12	**261**	**1.01106**	**10**	**97**	**0.99436**	**-**	**238**	**1.01339**	**10**	**80**	**0.99508**	**-**
	100	11	303	1.00839	0	117	-	1.00843	286	1.00941	0	101	-	1.00946
		12	**290**	**1.00995**	**2**	**133**	**0.99508**	**1.01004**	**269**	**1.01156**	**3**	**112**	**0.99533**	**1.01168**
		13	281	1.01155	10	140	0.99359	-	257	1.01383	10	109	0.99404	-

Table 6. Gaussian secrets (continued)

β	n	$\log(q)$	d_{2008}	δ_{2008}	Number of successes	Average time (min)	Average successful δ	Average failed δ	d_{2016}	δ_{2016}	Number of successes	Average time (min)	Average successful δ	Average failed δ
		12	319	1.00904	0	104	-	1.00912	301	1.01018	0	147	-	1.01024
	110	13	**309**	**1.01049**	**5**	**166**	**0.99425**	**1.01053**	**287**	**1.01215**	**10**	**170**	**0.99458**	**-**
		14	300	1.01196	10	203	0.99302	-	276	1.01417	10	193	0.99346	-
		13	337	1.00961	0	235	-	1.00965	317	1.01084	0	259	-	1.01091
	120	14	**327**	**1.01096**	**10**	**278**	**0.99372**	**-**	**305**	**1.01263**	**8**	**209**	**0.99389**	**1.01270**
		15	320	1.01232	10	321	0.99243	-	295	1.01446	10	139	0.99285	-
		13	365	1.00887	0	331	-	1.00890	347	1.00979	0	300	-	1.00985
35	130	14	**355**	**1.01011**	**2**	**341**	**0.99413**	**1.01014**	334	1.01139	0	286	-	1.01146
		15	346	1.01137	10	376	0.99312	-	**324**	**1.01303**	**10**	**245**	**0.99339**	**-**
		14	382	1.00939	0	438	-	1.00942	363	1.01038	0	468	-	1.01044
	140	15	**373**	**1.01055**	**2**	**463**	**0.99366**	**1.01059**	352	1.01186	0	424	-	1.01190
		16	365	1.01173	10	490	0.99274	-	**342**	**1.01337**	**10**	**398**	**0.99292**	**-**
		15	399	1.00985	0	526	-	1.00991	380	1.01088	0	756	-	1.01093
	150	16	**391**	**1.01095**	**3**	**580**	**0.99309**	**1.01101**	**370**	**1.01226**	**1**	**557**	**0.99331**	**1.01231**
		17	385	1.01205	10	698	0.99234	-	361	1.01366	10	487	0.99236	-
		6	207	1.00183	0	9	-	1.00403	233	1.00313	0	20	-	1.00318
	40	7	**170**	**1.00677**	**3**	**13**	**0.99979**	**1.00694**	155	1.00820	0	12	-	1.00835
		8	150	1.00998	10	12	0.99755	-	**113**	**1.01773**	**10**	**10**	**0.99967**	**-**
		7	213	1.00487	0	18	-	1.00550	212	1.00547	0	26	-	1.00555
	50	8	**187**	**1.00798**	**6**	**19**	**0.99849**	**1.00813**	**166**	**1.01020**	**5**	**35**	**0.99884**	**1.01032**
		9	171	1.01086	10	20	0.99619	-	136	1.01713	10	19	0.99788	-
		8	225	1.00664	0	41	-	1.00671	213	1.00740	0	31	-	1.00749
	60	9	**205**	**1.00904**	**8**	**45**	**0.99720**	**1.00912**	**180**	**1.01172**	**9**	**39**	**0.99748**	**1.01185**
		10	192	1.01148	10	47	0.99502	-	159	1.01679	10	32	0.99676	-
	70	9	239	1.00775	0	65	-	1.00781	221	1.00904	0	73	-	1.00914
		10	**223**	**1.00983**	**10**	**79**	**0.99618**	**-**	**197**	**1.01263**	**10**	**51**	**0.99644**	**-**
		9	273	1.00677	0	86	-	1.00682	261	1.00739	0	75	-	1.00747
	80	10	**255**	**1.00859**	**1**	**99**	**0.99653**	**1.00868**	234	1.01019	0	63	-	1.01032
		11	242	1.01049	10	102	0.99520	-	**215**	**1.01330**	**10**	**68**	**0.99588**	**-**
		10	287	1.00764	0	127	-	1.00769	271	1.00857	0	132	-	1.00863
	90	11	**272**	**1.00932**	**1**	**138**	**0.99573**	**1.00943**	**250**	**1.01110**	**2**	**127**	**0.99623**	**1.01117**
40		12	261	1.01106	10	169	0.99440	-	234	1.01382	10	129	0.99515	-
		11	303	1.00839	0	173	-	1.00843	284	1.00954	0	203	-	1.00960
	100	12	**290**	**1.00995**	**6**	**194**	**0.99498**	**1.01004**	**267**	**1.01182**	**5**	**166**	**0.99567**	**1.01186**
		13	281	1.01155	10	209	0.99357	-	253	1.01424	10	149	0.99419	-
	110	12	319	1.00904	0	251	-	1.00912	299	1.01034	0	285	-	1.01038
		13	**309**	**1.01049**	**10**	**312**	**0.99430**	**-**	**284**	**1.01242**	**10**	**255**	**0.99472**	**-**
	120	13	337	1.00961	0	346	-	1.00965	315	1.01102	0	344	-	1.01105
		14	**327**	**1.01096**	**10**	**403**	**0.99367**	**-**	**302**	**1.01289**	**10**	**359**	**0.99393**	**-**
		13	365	1.00887	0	409	-	1.00890	345	1.00990	0	345	-	1.00997
	130	14	**355**	**1.01011**	**1**	**541**	**0.99421**	**1.01014**	332	1.01158	0	433	-	1.01160
		15	346	1.01137	10	552	0.99310	-	**320**	**1.01329**	**10**	**368**	**0.99350**	**-**
		14	382	1.00939	0	597	-	1.00942	361	1.01051	0	485	-	1.01056
	140	15	**373**	**1.01055**	**4**	**606**	**0.99376**	**1.01059**	**349**	**1.01205**	**1**	**574**	**0.99384**	**1.01211**
		16	365	1.01173	10	656	0.99281	-	339	1.01363	10	537	0.99320	-
		15	399	1.00985	0	716	-	1.00991	378	1.01102	0	690	-	1.01105
	150	16	**391**	**1.01095**	**8**	**842**	**0.99320**	**1.01101**	**367**	**1.01246**	**2**	**717**	**0.99356**	**1.01251**
		17	385	1.01205	10	914	0.99233	-	358	1.01391	10	589	0.99239	-
		6	207	1.00183	0	90	-	1.00403	240	1.00294	0	118	-	1.00300
	40	7	**170**	**1.00677**	**4**	**158**	**1.00007**	**1.00694**	**152**	**1.00856**	**5**	**168**	**0.99977**	**1.00869**
		8	150	1.00998	10	149	0.99754	-	90	1.02778	10	42	1.00463	-
		7	213	1.00487	0	175	-	1.00550	213	1.00544	0	121	-	1.00550
	50	8	**187**	**1.00798**	**7**	**236**	**0.99854**	**1.00813**	**161**	**1.01079**	**10**	**248**	**0.99938**	**-**
		9	171	1.01086	10	300	0.99608	-	126	1.01990	10	107	0.99855	-
	60	8	225	1.00664	0	280	-	1.00671	211	1.00753	0	331	-	1.00763
		9	**205**	**1.00904**	**10**	**334**	**0.99723**	**-**	**175**	**1.01240**	**10**	**308**	**0.99839**	**-**
45		8	262	1.00569	0	367	-	1.00576	259	1.00585	0	521	-	1.00589
	70	9	**239**	**1.00775**	**1**	**410**	**0.99782**	**1.00781**	218	1.00929	0	347	-	1.00939
		10	223	1.00983	10	436	0.99620	-	**192**	**1.01327**	**10**	**441**	**0.99748**	**-**
	80	10	255	1.00859	0	542	-	1.00868	231	1.01049	0	615	-	1.01059
		11	**242**	**1.01049**	**10**	**656**	**0.99518**	**-**	**211**	**1.01391**	**10**	**593**	**0.99559**	**-**
		10	287	1.00764	0	753	-	1.00769	269	1.00871	0	587	-	1.00876
	90	11	**272**	**1.00932**	**4**	**804**	**0.99595**	**1.00943**	**246**	**1.01143**	**3**	**762**	**0.99634**	**1.01154**
		12	261	1.01106	10	908	0.99427	-	229	1.01439	10	870	0.99482	-
	100	11	303	1.00839	0	941	-	1.00843	281	1.00973	0	840	-	1.00980
		12	**290**	**1.00995**	**8**	**1080**	**0.99509**	**1.01004**	**263**	**1.01216**	**10**	**963**	**0.99534**	**-**

In Figures 1, 2, 3, we plot the values of δ for various blocksizes, β, against the lattice dimension for the binary, ternary and gaussian secret distributions respectively, with separate graphs for each blocksize. In each graph, we plot the values of the 2008 and 2016 estimates for δ against the dimension of the lattice, using blue dots for 2008 and blue crosses for 2016 predictions. For comparison, we plot Chen's estimate (2.5) which only depends on the blocksize, using a black line. In green and red, we plot the average *observed* values of δ for the instances where BKZ2.0 succeeds and fails. The green and red represent the successful and failed instances respectively. The dots and crosses represent attacks run with the dimensions calculated from the 2008 and 2016 estimates respectively. The data for the successful cases is obtained from the smallest value of $\log(q)$ where the attack succeeds, which are the rows in boldface in the tables, while the data for the failed cases is obtained from the largest value of $\log(q)$ where the attack does not succeed, which are the rows directly above those in boldface.

FIGURE 1. Comparing estimates (blue) for δ with observed δs from successful (green) and failed (red) attacks for binary secrets and block sizes $\beta = 30, 35, 40, 45$. The black line is Chen's estimate.

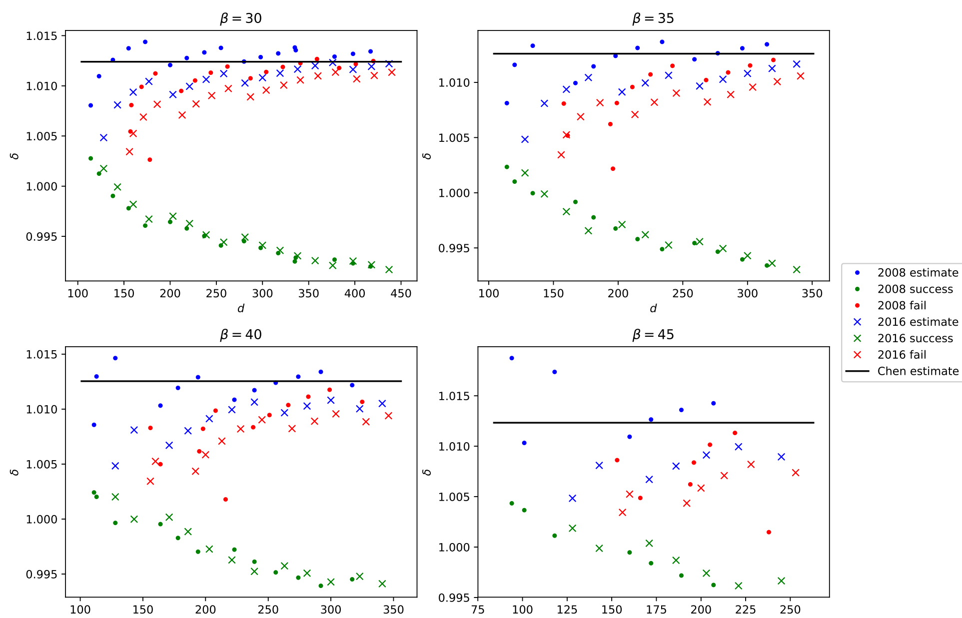


FIGURE 2. Plots of δ for ternary secrets and $\beta = 30, 35, 40, 45$.

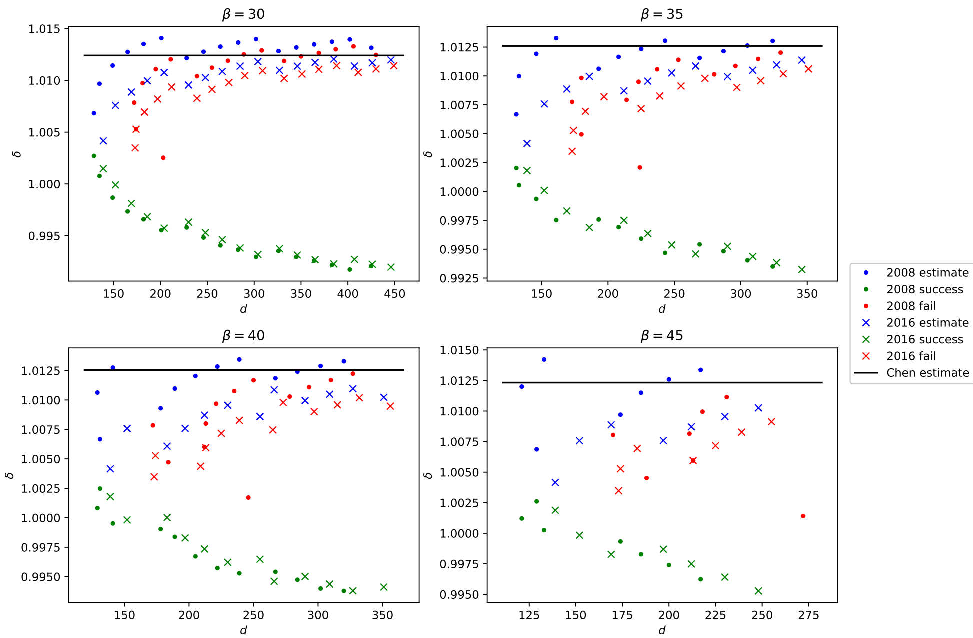
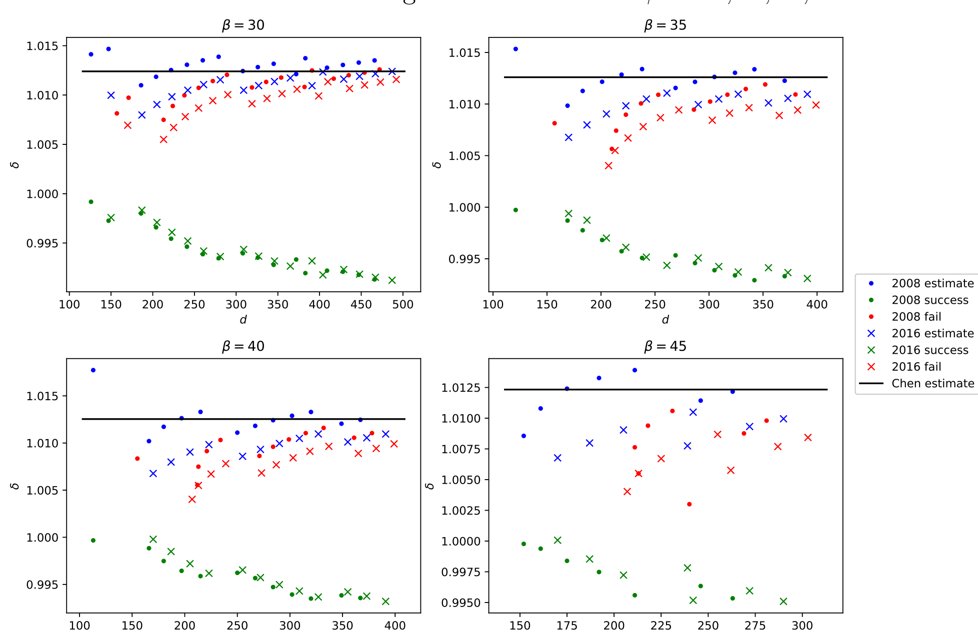
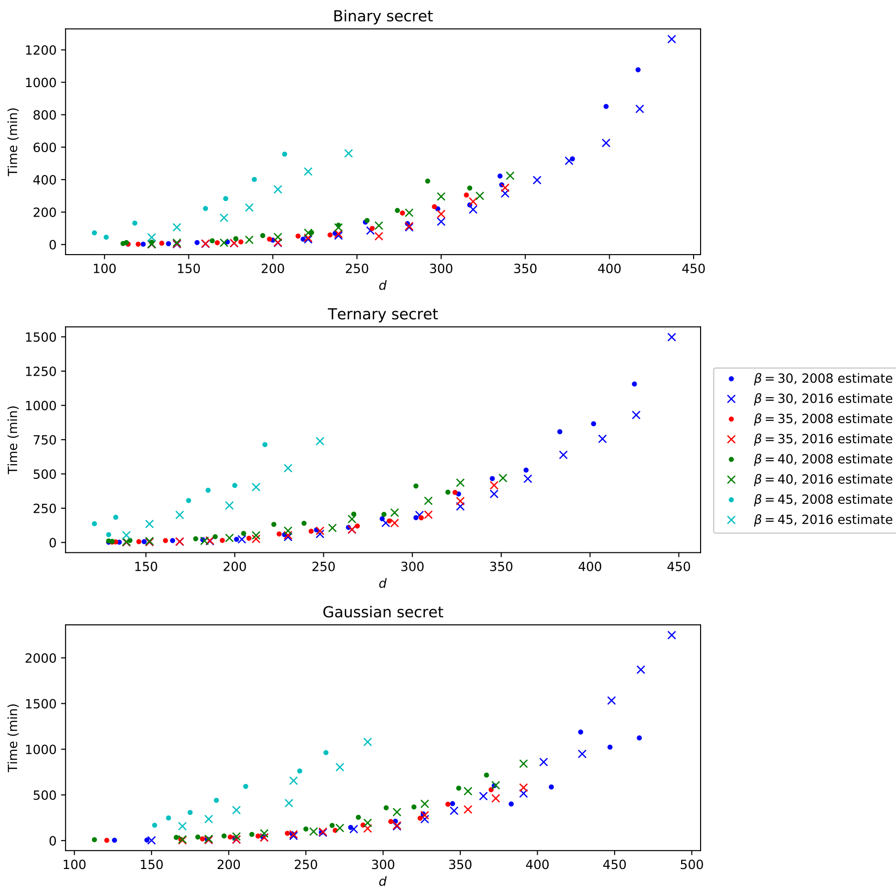


FIGURE 3. Plots of δ for gaussian secrets and $\beta = 30, 35, 40, 45$.



Due to the experimental nature of our work, we could only produce data for lattices of small dimensions, as the running time of BKZ2.0 grows exponentially with the parameters. We plot the average running times of BKZ2.0 for each set of parameters in Figure 4. In the plots, the dots and crosses represent attacks run with the dimensions calculated from the 2008 and 2016 estimates respectively. The blue, red, green and cyan represent blocksizes 30, 35, 40, 45 respectively.

FIGURE 4. Plots of running times in minutes. The dots and crosses represent attacks run with the dimensions calculated from the 2008 and 2016 estimates respectively. The blue, red, green and cyan represent blocksizes 30, 35, 40, 45 respectively.



4.2. **Results.** The main observation is that the experimental values of δ for successful instances decrease as the lattice dimension increases, whereas the 2008 and 2016 estimates show increasing trends which seem to approach Chen's estimate.

We observe that the experimental values of δ for *failed* instances closely follow the 2008 and 2016 estimates. The values of δ for failed instances are higher than for successful instances, which is expected since BKZ2.0 finds shorter vectors for the latter. Moreover, the values of δ for the successful instances decrease as the lattice dimension increases. In the cases where BKZ2.0 fails to recover the unique shortest vector, it recovers instead a vector with length close to the Gaussian Heuristic, and so the algorithm behaves like it would on a random lattice of the same dimension. In these cases, the experimental values of δ closely follow the 2008 and 2016 estimates. This indicates that the estimates accurately capture the behavior of BKZ2.0 on random lattices, but not on successful instances of uSVP.

We also observe that the success rates for the 2008 and 2016 estimates are comparable, although the 2008 estimate generally predicts higher lattice dimensions which lead to longer running times. The 2008 estimate also generally predicts higher values of δ than the 2016 estimate, for fixed lattice dimensions.

Additionally, for fixed n and β, the values of $\log q$ and d required to recover the secret is significantly higher for the cases where the secret is sampled from the discrete gaussian distribution, as compared to the binary and ternary distributions. The values for the binary and ternary distributions are comparable, though slightly higher for the ternary distribution. This indicates that gaussian secrets yield greater security levels, and would be recommended over binary or ternary secrets in practical applications. For all three secret distributions, the shortest vector has the same ℓ_2-norm, whereas the ℓ_1-norm is highest for the gaussian distribution, followed by the ternary and binary distributions. This indicates a trend of higher security level with increasing ℓ_1-norm, and it would be interesting to study this more systematically.

It is infeasible to run our experiments for $\beta \geq 50$ and $n > 100$ within reasonable times. For comparison, with blocksize 50, it takes about 19 hours to run the experiment with binary secrets for $n = 40$ and $\log q = 6$, as compared to an hour for blocksize 45 with the same parameters. It would be desirable to conduct longer experimental studies with higher blocksizes and dimensions, to simulate the parameters used in practical cryptosystems. Nevertheless, our work represents a first step towards a systematic experimental understanding of the success characteristics of BKZ2.0 on uSVP lattices, which we hope will motivate further studies on the topic.

4.3. **TU Darmstadt LWE Challenge.** Using the same experimental setup, we generate instances of the TU Darmstadt LWE challenges [BBG+16]. In the actual challenges, the secrets are sampled from uniform distributions on $\mathbb{Z}_q$; in our experiments we use instead the binary, ternary and gaussian secret distributions.

In the challenges, the discrete Gaussian error distributions have varying standard deviations $\sigma = \alpha q$, where α is a parameter. For each challenge, the parameters n, q, α are fixed. We generate instances with binary, ternary and Gaussian secrets, and we run the uSVP attack using the BKZ2.0 algorithm with blocksizes $\beta = 30, 35, 40, 45$. Due to resource limitations, we run only 3 trials for each set of parameters.

The data from our experiments are in Figures 5, 6, 7. In each figure, we plot a grid for each blocksize, where the columns are indexed by n and the rows by α. Each cell in the grid is colored based on the number of successful trials, where the colors red, orange, yellow and green indicate that the number of successful trials is 3, 2, 1 and 0 respectively. Moreover, the bottom diagonal of each divided cell indicates the 2008 estimate, while the top diagonal indicates the 2016 estimate.

We observe that there is a much higher success rate in solving the challenges for the binary and ternary secret distributions, as compared to the gaussian distribution. This indicates that gaussian secret distributions are more secure for practical applications. Moreover, our running times for the successful instances are significantly less than the records in the actual challenges, which use secrets from uniform distributions. Furthermore, we also observe that we are already able to attack all the solved LWE challenges in the online tables, for secrets sampled from the binary or ternary secret distributions. This indicates that uniform secrets offer much higher security than the secret distributions that we consider, and it would be promising to study the case of uniform secrets in our experimental framework, as a potential follow-up to this work.

FIGURE 5. TU Darmstadt LWE challenges for binary secrets. Each cell is colored based on the number of successful trials, where the colors red, orange, yellow and green indicate that the number of successful trials is 3, 2, 1 and 0 respectively. The bottom diagonal of each divided cell indicates the 2008 estimate, while the top diagonal indicates the 2016 estimate.

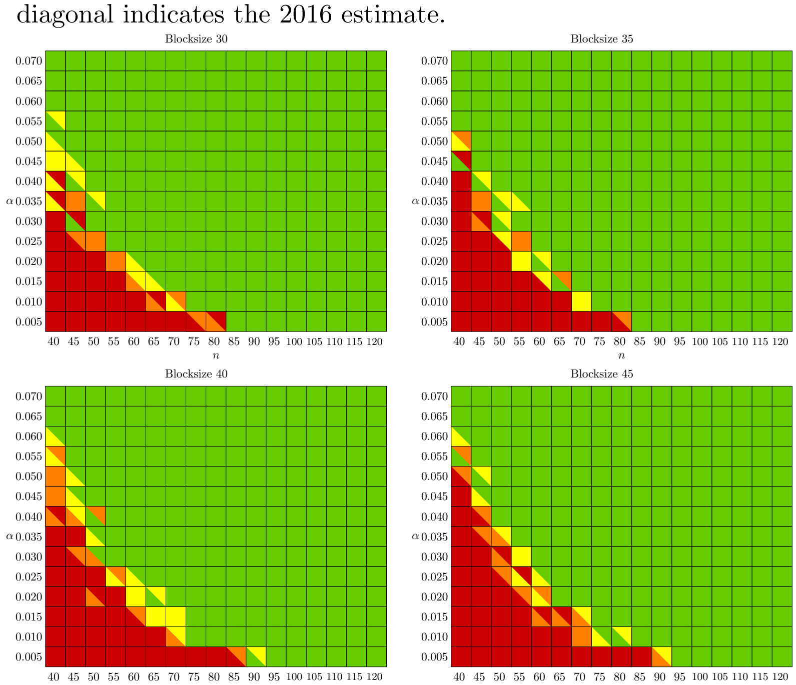
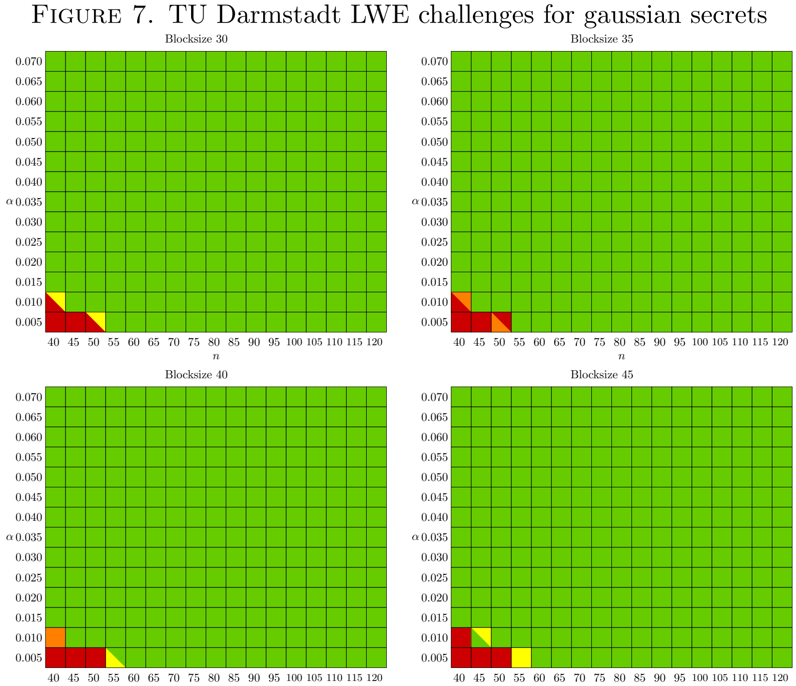


FIGURE 6. TU Darmstadt LWE challenges for ternary secrets

FIGURE 7. TU Darmstadt LWE challenges for gaussian secrets



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