The Topology of Common Belief*

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Abstract. We study the modal logic $\mathbf{K4_2^C}$ of common belief for normal agents. We study Kripke completeness and show that the logic has tree model property. As a main result we prove that $\mathbf{K4_2^C}$ is the modal logic of all T_D-intersection closed, bi-topological spaces with derived set interpretation of modalities. Based on the splitting translation we discuss connections with $\mathbf{S4_2^C}$, the logic of common knowledge.

1 Introduction

In logics for knowledge representation and reasoning, the study of epistemic and doxastic properties of agents with certain, intuitively acceptable, restrictions on their knowledge and belief is a well-developed area. Smullyan [10] discusses various types of agents based on properties of belief. In his terminology, an agent whose belief satisfies the modal axiom $(4) : \Box p \to \Box\Box p$, translated as 'If the agent believes p, then he believes that he believes p', is called a *normal agent*. $\mathbf{K4}$ is the modal logic which formalizes the belief behavior of normal agents. This generalizes the classical doxastic system $\mathbf{KD45}$ in the same way as $\mathbf{S4}$ generalizes the epistemic logic $\mathbf{S5}$, by dropping some restrictions on the properties of an agent.

Agreement technologies is a newly emerging domain where iterative concepts of belief and knowledge of agents are of special interest. To achieve successful communication and agreement it is important for agents to reason about themselves and what others know or believe. Among the more interesting cases are the notions of common knowledge and common belief. We denote the operators for common knowledge and common belief by C_K and C_B respectively. We have: $C_K\varphi$ iff φ is common knowledge in the group K and $C_B\varphi$ iff φ is a common belief in the group B.

Following the analysis of common knowledge as originally defined by Lewis [19], this concept has been extensively studied from various perspectives in philosophy [20], [15], game theory [2], artificial intelligence [4], modal logic [17], [18], [14] etc. Theories of common belief are less well-developed though some approaches can be found in [3,4,27]. The present paper is devoted to a study of the common belief of normal agents. Our aim is to extend two previous lines of work. Earlier, in [24,25,26] we have examined several extensions of the modal logic $\mathbf{wK4}$ that form interesting doxastic logics different from $\mathbf{KD45}$. Our main interests were related to the idea of minimal belief, non-monotonic reasoning about beliefs, topological interpretations and in each case the embedding relations between epistemic and doxastic logics, i.e. translations between knowledge and belief operators. In this previous work we considered only single

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agent systems. A second point of departure is provided by the work of van Benthem and Sarenac [13], who showed how a topological semantics for logics of common knowledge may be useful for modeling and distinguishing different concepts. A key idea here is that the knowledge of different agents is represented by different topologies over a set X. Various ways to merge that knowledge can be obtained via different modes of combining logics and topological models. [13] considers for example the fusion logic $\mathbf{S4} \circ \mathbf{S4}$ and product topologies that are complete for the common knowledge logic $\mathbf{S4_2^C}$ of [16].

In light of [16,13] and our previous work several natural questions emerge that we address here. In summary the main contributions of the paper are:

1. We define a logic $\mathbf{K4_2^C}$ of common belief for normal agents and prove its completeness for a Kripke, relational semantics. We show it has the finite model property and the tree model property.
2. We study the topological semantics for $\mathbf{K4_2^C}$ and show completeness for intersection topologies. Specifically we show that $\mathbf{K4_2^C}$ is the modal logic of all T_D-intersection closed, bi-topological spaces with a derived set interpretation of modalities.
3. Belief under the topological interpretation of $\mathbf{K4_2^C}$ is understood via colimits and common belief in terms of colimits in the intersection topology. From 2 we derive a topological condition for common belief in terms of colimits that is very similar to the corresponding condition that defines common knowledge in the modal μ-calculus and is discussed at some length in [13].
4. We show how the common knowledge logic $\mathbf{S4_2^C}$ can be embedded in $\mathbf{K4_2^C}$ via the splitting translation that maps $C_K p$ into $p \wedge C_B p$.

1.1 Common belief and the topological interpretation

As stated, we focus on the common belief of normal agents, and for ease of exposition we restrict ourselves to the two agent case. We thus consider two agents whose individual beliefs satisfy the axioms of $\mathbf{K4}$. In other respects we adopt the main principles of the logic of common knowledge, $\mathbf{S4_2^C}$. This can be seen as a formalization of the idea that common knowledge is equivalent to an infinite conjunction of iterated individual knowledge: $\varphi \wedge \Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1\Box_1\varphi \wedge \Box_1\Box_2\varphi \wedge \Box_2\Box_1\varphi \wedge \Box_2\Box_2\varphi \wedge \Box_1\Box_1\Box_1\varphi \wedge \Box_1\Box_1\Box_2\varphi$... Later we shall see that a variation of this formula is 'true' for common belief under the relational semantics. We shall also show that the topological semantics for $\mathbf{K4_2^C}$ is compatible with the idea of common belief as a fixpoint *equilibrium*, a notion used by Barwise [20] to describe common knowledge that can be captured by an expression of the modal μ-calculus.

Our approach to providing a topological semantics follows the work of Esakia [22]. Notice that under the topological interpretation of $\Box$ as a knowledge operator, eg. in [13], $\Box\varphi$ refers to the topological *interior* of the points assigned to φ. In the case of a doxastic logic like $\mathbf{K4}$ the topological interpretation is different. It is perhaps simpler to state it for the $\Diamond$ operator. Following McKinsey and Tarski [12], the idea is to treat $\Diamond\varphi$ as the *derivative* of the set φ in the topological space. Esakia showed that under this interpretation $\mathbf{wK4}$ is the modal logic of all topological spaces. $\mathbf{K4}$ is an extension of $\mathbf{wK4}$ and is characterized in this semantics by the class of all T_D-spaces [21]. By combining the ideas and results from [13] and [22], we obtain a derived set semantics for the logic of common belief based on bi-topological spaces, where the modality for common belief operates on the intersection of the two topologies. As a main result, we can prove that $\mathbf{K4_2^C}$ is sound and complete with respect to the special subclass of all bi-topological T_D-spaces.

2 Logic of Common Belief

We turn to the syntax and Kripke semantics of the logic $\mathbf{K4_2^C}$. The interpretation of common belief operator $C_{_B}$ on bi-relational Kripke frames is similar to the interpretation of the common knowledge operator $C_{_K}$, and is based on the notion of transitive closure of a relation. In this section we show that the logic $\mathbf{K4_2^C}$ is sound and complete with respect to the class of all bi-relational transitive Kripke structures. The proof is a slight modification of the completeness proof for the logic $\mathbf{S4_2^C}$ given in [16] therefore we only sketch the essential parts where the difference shows up. Additionally we show that every non-theorem of $\mathbf{K4_2^C}$ can be falsified on an infinite, irreflexive, bi-transitive tree.

2.1 Iterative common belief

There are different notions of common belief [20]. Let us mention common belief as an infinite conjunction of nested beliefs and common belief as an equilibrium. Under the former idea, a proposition p is a common belief of two agents if: agent-1 believes that p and agent-2 two believes that p and agent-1 believes that agent-2 believes that p and agent-2 believes that agent-1 believes that p etc., where all possible finite mixtures occur. If we formalize this idea in a modal language with belief operators $\Box_1$ and $\Box_2$ for each agent respectively, then we arrive at the following concept of a common belief operator $C^{\omega}_{_B}$.

$$\begin{aligned}
&C^0_{_B}p = \Box_1 p \wedge \Box_2 p;\\
&C^{n+1}_{_B}p = \Box_1 C^n_{_B}p \wedge \Box_2 C^n_{_B}p;\\
&C^{\omega}_{_B}p = \textstyle\bigwedge_{n\in\omega} C^n_{_B}p.
\end{aligned}$$

$C^{\omega}_{_B}$ exactly formalizes the intuition behind the former idea of common belief. However, since $C^{\omega}_{_B}$ is an infinite intersection, it cannot be expressed as an ordinary formula of modal logic and hence studied in the usual approaches to standard modal logic. Nevertheless it turns out that we can capture the infinitary behavior of $C^{\omega}_{_B}$ in a finitary sense. This idea is made more precise via the modal logic $\mathbf{K4_2^C}$.

2.2 Syntax

Throughout we work in the modal language $\mathcal{L}_{\mathcal{C}}$ with an infinite set $Prop$ of propositional letters and symbols $\wedge, \neg, \Box_1, \Box_2, C_{_B}$. The set of formulas $Form$ is constructed in a standard way: $Prop \subseteq Form$. If $\alpha, \beta \in Form$ then $\neg\alpha, \alpha\wedge\beta, \Box_1\alpha, \Box_2\alpha, C_{_B}\alpha \in$

$Form$. We will use standard abbreviations for disjunction and implication, $\alpha \vee \beta \equiv \neg(\neg\alpha \wedge \neg\beta)$ and $\alpha \to \beta \equiv \neg\alpha \vee \beta$.

• The axioms of the logic $\mathbf{K4_2^C}$ are all classical tautologies, each box satisfies all $\mathbf{K4}$ axioms, ie. we have: $(K)\ \Box_i(p \to q) \to (\Box_i p \to \Box_i q)$, $(4)\ \Box_i p \to \Box_i \Box_i p$, for each $i \in \{1, 2\}$ and in addition we have the equilibrium axiom for the common belief operator:

$$(equi) : C_{\scriptscriptstyle B} p \leftrightarrow \Box_1 p \wedge \Box_2 p \wedge \Box_1 C_{\scriptscriptstyle B} p \wedge \Box_2 C_{\scriptscriptstyle B} p.$$

• The rules of inference are: Modus-Ponens, Substitution, Necessitation for $\Box_1$ and $\Box_2$ and the induction rule for the common belief operator:

$$(ind) : \frac{\vdash \varphi \to \Box_1(\varphi \wedge \psi) \wedge \Box_2(\varphi \wedge \psi)}{\vdash \varphi \to C_{\scriptscriptstyle B} \psi}$$

where φ and ψ are arbitrary formulas of the language.

2.3 Kripke Semantics

The Kripke semantics for the modal logic $\mathbf{K4_2^C}$ is provided by transitive, bi-relational Kripke frames. The triple (W, R_1, R_2), with W an arbitrary set and $R_i \subseteq W \times W$ where $i \in \{1, 2\}$, is a *bi-transitive Kripke frame* if both R_1 and R_2 are transitive relations. A quadruple (W, R_1, R_2, V) is a bi-transitive Kripke model if (W, R_1, R_2) is a bi-transitive Kripke frame and $V : Prop \to P(W)$ is a valuation function. Observe that we only have two relations, which give a semantics for $\Box_1$ and $\Box_2$. To interpret the common belief operator, $C_{\scriptscriptstyle B}$, we construct a new relation, which is a transitive closure of the union of R_1 and R_2.

Definition 1. *The transitive closure R^+ of a relation R is defined as the least transitive relation containing the relation R.*

Two points x and y are related by the transitive closure of the relation if there exists a finite path $\langle x_1, .., x_n \rangle$ starting at x and ending at y.

Definition 2. *For a given bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ the satisfaction of a formula at a point $w \in W$ is defined inductively as follows:*
$w \Vdash p$ iff $w \in V(p)$,
$w \Vdash \alpha \wedge \beta$ iff $w \Vdash \alpha$ and $w \Vdash \beta$,
$w \Vdash \neg\alpha$ iff $w \nVdash \alpha$,
$w \Vdash \Box_i \varphi$ iff $(\forall v)(w R_i v \Rightarrow v \Vdash \varphi)$,
$w \Vdash C_{\scriptscriptstyle B} \varphi$ iff $(\forall v)(w (R_1 \cup R_2)^+ v \Rightarrow v \Vdash \varphi)$.
A formula α is valid in a model $\mathcal{M}$, in symbols $\mathcal{M} \Vdash \alpha$, if for every point $w \in W$ we have $w \Vdash \alpha$. α is valid in a bi-relational frame $\mathcal{F} = (W, R_1, R_2)$, in symbols $\mathcal{F} \Vdash \alpha$, iff α is valid in every model $\mathcal{M} = (\mathcal{F}, V)$ based on the frame. α is valid in a class of bi-relational frames K if for every frame $\mathcal{F} \in K$ we have $\mathcal{F} \Vdash \alpha$.

Proposition 1. *(Completeness) Modal logic $\mathbf{K4_2^C}$ is sound and complete with respect to the class of all finite, bi-transitive Kripke frames.*

Proof. (Sketch) The proof follows the pattern of [16] for the logic $\mathbf{S4_2^C}$. The only difference appears when defining the canonical relation which may not be just transitive if defined in the same way as in [16]. Therefore following [5] we define the relations R_1 and R_2 on W in the following way: For every maximal consistent sets of formulas $\Gamma, \Gamma' \in W$ we define $\Gamma R_x \Gamma'$ iff $(\forall\alpha)(\Box_x\alpha \in \Gamma \Rightarrow \Gamma' \vdash \alpha \wedge \Box_x\alpha)$, where $x \in \{1, 2\}$.

According to proposition 1 every non-theorem of $\mathbf{K4_2^C}$ is falsified on a finite, bi-transitive frame. The following theorem shows that every non-theorem of $\mathbf{K4_2^C}$ can be falsified on a frame (W^t, R_1^t, R_2^t, V^t), where for each $k \in \{1, 2\}$ the pair (W^t, R_k^t) is a transitive tree.

Definition 3. *A frame* (W, R) *is called a* tree *if:*
1) it is rooted, ie. there is a unique point (the root) $r \in W$ *such that for every* $v \in W$ *it holds that* $v \neq r \Rightarrow rR^+v$*,*
2) every element distinct from r *has a unique immediate predecessor; that is, for every* $v \neq r$ *there is a unique* v' *such that* $v'Rv$ *and for every* v'' *we have that* $v''Rv \Rightarrow v''Rv'$*,*
3) R *is acyclic; that is, for every* $v \in W$ *it is not the case that* vR^+v*.*
If in addition R *is transitive, ie.* $R = R^+$*, then* (W, R) *is called a transitive tree.*

Theorem 1. *The modal logic* $\mathbf{K4_2^C}$ *has tree model property.*

Proof. (Sketch) We start with a bi-relational countermodel $\mathcal{M} = (W, R_1, R_2, V)$ for the formula φ. The proof follows a standard unravelling technique [1]. As a result we get a model $\mathcal{M}^t = (W^t, R_1^t, R_2^t, V^t)$, where (W^t, R_k^t) is a tree for each $k \in \{1, 2\}$ and the valuation V^t is defined by reflecting the valuation V of the original countermodel $\mathcal{M}$. Additionally $\mathcal{M}^t \nVdash \varphi$ as far as $\mathcal{M}$ is a bounded morphic image of $\mathcal{M}^t$ and the bounded morphism extends between models $\mathcal{M}^t = (W^t, R_1^t, R_2^t, (R_1^t \cup R_2^t)^+, V^t)$ and $\mathcal{M} = (W, R_1, R_2, (R_1 \cup R_2)^+, V)$.

Note 1. Observe that the relation $(R_1^t \cup R_2^t)^+$ does not contain cycles and in particular it is irreflexive.

The mains reason for introducing $\mathbf{K4_2^C}$ was to mimic the infinitary operator C_B^ω by finitary C_B. Though we cannot claim that on a logical level C_B and C_B^ω are equivalent, we can establish a semantical equivalence, in particular on Kripke structures.

Theorem 2. *For any transitive bi-relational Kripke model* $\mathcal{M} = (W, R_1, R_2, V)$ *and point* w*:* $\mathcal{M}, w \Vdash C_B\varphi$ *iff* $\mathcal{M}, w \Vdash C_B^\omega$*.*

Proof. The proof follows easily from Definitions 1 and 2.

2.4 Common belief as equilibrium

We mentioned that common belief can also be understood as an equilibrium concept[1]. On Kripke structures the equilibrium conception coincides with common belief by infinite iteration, while in general the equilibrium conception has a much closer connection

[1] For the remainder of this section and later on for Theorem 9 we assume some familiarity with the modal μ-calculus. Lack of space hinders a fuller treatment, however for more details on the modal μ-calculus we refer to [1][part 3, chapter 4]; see also the discussion in [13].

to the logic $\mathbf{K4_2^C}$. It can be formalized in the modal μ-calculus in the following way:

$$C_\nu \varphi = \nu.p(\Box_1 \varphi \wedge \Box_2 \varphi \wedge \Box_1 p \wedge \Box_2 p).$$

The greatest fixpoint ν is defined as the fixpoint of a descending approximation sequence defined over the ordinals. Denote by $|\varphi|$ the truth set of φ in the appropriate model $\mathcal{M}$ where evaluation occurs:

$$|C_\nu^0 \varphi| = |\Box_1 \varphi \wedge \Box_2 \varphi|;$$
$$|C_\nu^{k+1} \varphi| = |\Box_1 \varphi \wedge \Box_2 \varphi \wedge \Box_1 C_\nu^k \varphi \wedge \Box_2 C_\nu^k \varphi|;$$
$$|C_\nu^\lambda \varphi| = |\bigcap_{k<\lambda} C_\nu^k \varphi|, \text{ for } \lambda \text{ a limit ordinal.}$$

We obtain $|C_\nu \varphi| = |C_\nu^\gamma \varphi|$, where γ is a least ordinal for which the approximation procedure halts: ie. $|C_\nu^\gamma \varphi| = |C_\nu^{\gamma+1} \varphi|$. Halting is guaranteed because the occurrence of the propositional variable p in operator $F(p)$, where $F(p) = \Box_1 \varphi \wedge \Box_2 \varphi \wedge \Box_1 p \wedge \Box_2 p$, is positive. Hence by the Knaster-Tarski theorem the sequence will always reach a greatest fixpoint. Then the semantics of the operator C_ν is defined in the following way:

$$\mathcal{M}, w \Vdash C_\nu \varphi \text{ iff } w \in |C_\nu^\gamma \varphi|$$

In general this procedure may take more than ω steps, but in case of Kripke structures the situation is simpler. The following property relates the different operators on Kripke models.

Theorem 3. *For every bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ and a point $w \in W$ the following condition holds: $\mathcal{M}, w \Vdash C_B^\omega \varphi$ iff $\mathcal{M}, w \Vdash C_\nu \varphi$.*

Proof. Observe that we can rewrite $C_B^\omega \varphi = \Box_1 \varphi \wedge \Box_2 \varphi \wedge \Box_1 \Box_1 \varphi \wedge \Box_1 \Box_2 \varphi \wedge \Box_2 \Box_1 \varphi \wedge \Box_2 \Box_2 \varphi \wedge \Box_1 \Box_1 \Box_1 \varphi \wedge \Box_1 \Box_1 \Box_2 \varphi$... in the following way: $\Box_1 \varphi \wedge \Box_2 \varphi \wedge \Box_1 (\Box_1 \varphi \wedge \Box_2 \varphi) \wedge \Box_2 (\Box_1 \varphi \wedge \Box_2 \varphi) \wedge$ ... Hence $|C_B^\omega \varphi| = |C_\nu^\omega \varphi|$. It is known that on Kripke structures stabilization process does not need more than ω steps [13] i.e. $|C_\nu \varphi| = |C_\nu^\omega \varphi|$. Hence $w \Vdash C_\nu \varphi$ iff $w \Vdash C_B^\omega \varphi$

It follows that on transitive bi-relational Kripke structures the three operators C_B, C_B^ω and C_ν coincide.

3 Topological Semantics

The idea of a derived set topological semantics originates with the McKinsey-Tarski paper [12]. This idea was taken further in [22]. The following works contain some important results in this direction: [21], [8], [6], [7]. The derived set topological semantics for $\mathbf{K4_2^C}$ is provided by the class of all bi-topological spaces. In the same way, as it is done in [13] for the common knowledge operator, we interpret the common belief operator on the intersection topology. On the other hand, different from C_K, for which the semantics is given using interior of the intersection of the two topologies, we provide the semantics of $C_B \varphi$ as a set of all colimits of $|\varphi|$ in the intersection topology. As a main result we prove the soundness and completeness of the logic $\mathbf{K4_2^C}$ with respect to the class of all T_D-*intersection closed*, bi-topological spaces where each topology satisfies the T_D separation axiom. We start with the basic definitions.

Definition 4. *A pair (X, Ω) is called a topological space if X is a set and Ω is a collection of subsets of X with the following properties:*
1) $X, \varnothing \in \Omega$,
2) $A, B \in \Omega$ implies $A \cap B \in \Omega$,
3) $A_i \in \Omega$ implies $\bigcup A_i \in \Omega$.
Elements of Ω are called opens or open sets of the topological space.

Definition 5. *A topological space (X, Ω) is called an Alexandroff space if an arbitrary intersection of opens is open, that is $A_i \in \Omega$ implies $\bigcap A_i \in \Omega$. (X, Ω) is called a T_D-space if every point $x \in X$ can be represented as an intersection of some open set A and some closed set B.*

We now define the colimit operator (or the set of all colimit points [11]) of a set in a topological space. This is needed to give the semantics of modal formulas in an arbitrary topological space.

Definition 6. *Given a topological space (X, Ω) and a set $A \subseteq X$ we will say that $x \in X$ is a colimit point of A if there exists an open neighborhood U_x of x such that $U_x - \{x\} \subseteq A$. The set of all colimit points of A will be denoted by $\tau(A)$ and will be called the colimit set of A.*

The colimit set provides a semantics for the box modality, consequently the semantics for diamond is provided by the dual of the colimit set, which is called the *derived* set. The derived set of A is denoted by $der(A)$. So we have $\tau(A) = X - der(X - A)$. Below we list some properties of the colimit operator.

Fact 4 *[11,23] For a given topological space (X, Ω) the following properties hold:*
1) $Int(A) = \tau(A) \cap A \subseteq \tau\tau(A)$, where Int denotes the interior operator,
2) $\tau(X) = X$ and $\tau(A \cap B) = \tau(A) \cap \tau(B)$,
3) If Ω is a T_d-space then $\tau(A) \subseteq \tau\tau(A)$,
4) If $\Omega_1 \subseteq \Omega_2$ then $\tau_1(A) \subseteq \tau_2(A)$ where τ_i, $i \in \{1, 2\}$ is a colimit operator of the corresponding topology Ω_i.

The following links T_D-spaces and irreflexive transitive relational structures. This result is a special case of a more general correspondence between weakly-transitive and irreflexive relational structures and all *Alexandroff* spaces [22].

Fact 5 *([23]) There is a one-to-one correspondence between Alexandroff, T_D-spaces and transitive, irreflexive relational structures.*

Let us briefly describe the correspondence. We first introduce the downset operator. Let (X, R) be a Kripke frame. The downset operator R^{-1} is defined in the following way: for any $A \subseteq X$ we set $R^{-1}(A) := \{x|(\exists y)(y \in A \wedge xRy)\}$. Now if we are given an irreflexive, transitive order (X, R) it is possible to prove that the downset operator R^{-1} satisfies all the properties of the topological derivative operator for T_D-spaces. Hence we get a T_D-space (X, Ω_R), where Ω_R is the topology obtained from the derivative operator R^{-1}. Conversely with every *Alexandroff* T_D-space (X, Ω), one can associate an irreflexive and transitive relational structure (X, R_Ω), where $xR_\Omega y$ iff $x \in der(\{y\})$. Moreover we have that (X, Ω_{R_Ω}) is homeomorphic to (X, Ω) and (X, R_{Ω_R}) is order isomorphic to (X, R).

Fact 6 *[23] The set A is open in (X, Ω_R) iff $x \in A$ implies that the implication $(xRy \Rightarrow y \in A)$ holds for every $y \in X$.*

This correspondence can be directly generalized to Kripke frames with more than one transitive and irreflexive relation. Of course then we will have one Alexandroff T_D-space for each irreflexive and transitive order. Below we prove the proposition which builds a bridge between Kripke and topological semantics for $\mathbf{K4_2^C}$.

Proposition 2. *If R_1 and R_2 are two irreflexive and transitive orders on X and $(R_1 \cup R_2)^+$ is also irreflexive and transitive, then $\Omega_{(R_1 \cup R_2)^+} \cong \Omega_{R_1} \cap \Omega_{R_2}$.*

Before starting the proof, observe that $(R_1 \cup R_2)^+$ may not be irreflexive even if both R_1 and R_2 are. For example: Let $X = \{x, y\}$ and $R_1 = \{(x, y)\}$ and $R_2 = \{(y, x)\}$ then $(R_1 \cup R_2)^+ = \{(x, y), (y, x), (x, x), (y, y)\}$. On the topological side this example shows that T_D-spaces do not form a lattice. That is why in Proposition 2 we require $(R_1 \cup R_2)^+$ to be a irreflexive and transitive.

Proof. Assume that $A \in \Omega_{(R_1 \cup R_2)^+}$. By Fact 6 this means that if $x \in A$ then for every y such that $x(R_1 \cup R_2)^+ y$ it holds that $y \in A$. Since $R_i \subseteq (R_1 \cup R_2)^+$ for each $i \in \{1, 2\}$, it holds that $xR_1y \Rightarrow y \in A$ and $xR_2y \Rightarrow y \in A$ for every $y \in X$. Hence $A \in \Omega_1 \cap \Omega_2$ according to Fact 6.

Conversely assume $A \in \Omega_1 \cap \Omega_2$. This means that $x \in A \Rightarrow (x(R_1 \cup R_2)y \Rightarrow y \in A)$. Now take arbitrary y such that $x(R_1 \cup R_2)^+ y$. By definition this means that there is a $(R_1 \cup R_2)$-path $\langle x_1, x_2, ..x_n\rangle$ starting at x going to y. But this means that each member of this path is in A because A is open in the intersection of the two topologies. Hence $y \in A$ and hence $A \in \Omega_{(R_1 \cup R_2)^+}$

Next we give a definition of the satisfaction relation of modal formulas in the derived set topological semantics. Observe that this definition is given in a standard modal language ie., without the common belief operator. Recall that a topological model is a tuple $\mathcal{M} = (W, \Omega, V)$ where $V : Prop \to P(W)$ is a valuation function.

Definition 7. *The satisfaction of a modal formula in a topological model $\mathcal{M} = (W, \Omega, V)$ at a point $w \in W$ is defined in the following way:*

- *$\mathcal{M}, w \Vdash p$ iff $w \in V(p)$,*
- *Boolean cases are standard,*
- *$\mathcal{M}, w \Vdash \Box\varphi$ iff $w \in \tau(V(\varphi))$, where τ is a colimit operator of Ω.*

Fact 7 *[23] The correspondence mentioned in the Fact 5 preserves the truth of modal formulas, ie. $(W, R, V), x \Vdash \alpha$ iff $(W, \Omega_R, V), x \Vdash \alpha$.*

Note that in Fact 7, the symbol $\Vdash$ on the left hand side denotes the satisfaction relation on Kripke models, while on the right hand side it denotes the satisfaction relation on topological frames in the derived set semantics. Now we extend the satisfaction relation to the language with the common belief operator.

Definition 8. *The satisfaction of a modal formula on a bi-topological model* $\mathcal{M} = (W, \Omega_1, \Omega_2, V)$ *at a point* $w \in W$ *is defined in the following way:*
$\mathcal{M}, w \Vdash p$ *iff* $w \in V(p)$*,*
$\mathcal{M}, w \Vdash \alpha \wedge \beta$ *iff* $\mathcal{M}, w \Vdash \alpha$ *and* $\mathcal{M}, w \Vdash \beta$*,*
$\mathcal{M}, w \Vdash \neg\alpha$ *iff* $\mathcal{M}, w \nVdash \alpha$*,*
$\mathcal{M}, w \Vdash \Box_i\varphi$ *iff* $w \in \tau_i(V(\varphi))$*, where* τ_i *is a colimit operator of* Ω_i*,* $i \in \{1, 2\}$*,*
$\mathcal{M}, w \Vdash C_B\varphi$ *iff* $w \in \tau_{1\wedge 2}(V(\varphi))$*, where* $\tau_{1\wedge 2}$ *is a colimit operator in* $\Omega_1 \cap \Omega_2$*.*

As an immediate corollary of the proposition 2 and a many-modal version of the Fact 7, we get the following proposition.

Proposition 3. *If* R_1 *and* R_2 *are two irreflexive and transitive orders and* $(R_1 \cup R_2)^+$ *is also topological then for every formula* α *in* $\mathbf{K4_2^C}$ *the following holds:*

$$(W, R_1, R_2, V), x \Vdash \alpha \text{ iff } (W, \Omega_{R_1}, \Omega_{R_2}, V), x \Vdash \alpha.$$

Now it is clear that we can reduce the topological completeness problem to Kripke completeness if for every non-theorem $\mathbf{K4_2^C} \nvdash \varphi$ we can find a bi-relational topological counter-model (W, R_1, R_2, V) with $(R_1 \cup R_2)^+$ being also a topological relation.

Definition 9. *The triple* (X, Ω_1, Ω_2) *is a* T_D*-intersection closed bi-topological space if each of the topologies* Ω_1*,* Ω_2 *and* $\Omega_1 \cap \Omega_2$*, satisfies the* T_D*-separation axiom.*

Theorem 8. $\mathbf{K4_2^C}$ *is sound and complete with respect to the class of all* T_D*-intersection closed, bi-topological, Alexandroff spaces.*

Proof. (Soundness) Take an arbitrary T_D-intersection closed, bi-topological model $\mathcal{M} = (X, \Omega_1, \Omega_2, V)$. From 2) and 3) of Fact 4 it follows that $K4$-axioms are valid for each box. Let us show that at each point $x \in X$, the equilibrium axiom is satisfied. Assume that $\mathcal{M}, x \Vdash C_Bp$. Hence by Definition 8 we have $x \in \tau_{1\wedge 2}|p|$. By 4) of Fact 4 we get $x \in \tau_1|p|$ and $x \in \tau_2|p|$. By 3) we have $\tau_{1\wedge 2}|p| \subseteq \tau_{1\wedge 2}\tau_{1\wedge 2}|p| \subseteq \tau_1\tau_{1\wedge 2}|p|$. Analogously $\tau_{1\wedge 2}|p| \subseteq \tau_2\tau_{1\wedge 2}|p|$. Hence we have $x \Vdash \Box_1 p \wedge \Box_2 p \wedge \Box_1 C_B p \wedge \Box_2 C_B p$.

For the other direction assume that $x \in \tau_1\tau_{1\wedge 2}|p| \cap \tau_1|p| \cap \tau_2\tau_{1\wedge 2}|p| \cap \tau_2|p|$. By 2) of Fact 4 we get $x \in \tau_1(\tau_{1\wedge 2}|p| \cap |p|) \cap \tau_2(\tau_{1\wedge 2}|p| \cap |p|)$. By 1) of Fact 4 we conclude $x \in \tau_1(Int_{1\wedge 2}|p|) \cap \tau_2(Int_{1\wedge 2}|p|)$, where $Int_{1\wedge 2}$ denotes the interior operator in the intersection topology. By definition of colimit there exists $U_x^1 \in \Omega_1$ such that $x \in U_x^1$ and $U_x^1 - \{x\} \subseteq Int_{1\wedge 2}|p|$ and there exists $U_x^2 \in \Omega_2$ such that $x \in U_x^2$ and $U_x^2 - \{x\} \subseteq Int_{1\wedge 2}|p|$. Hence $(U_x^1 \cup U_x^2) - \{x\} \subseteq Int_{1\wedge 2}|p|$. Let us show that $Int_{1\wedge 2}|p| \cup \{x\}$ is open in $\Omega_1 \cap \Omega_2$. Since $U_x^1 \in \Omega_1$ and $Int_{1\wedge 2}|p| \in \Omega_1$ we have $U_x^1 \cup Int_{1\wedge 2}|p| = Int_{1\wedge 2}|p| \cup \{x\} \in \Omega_1$. Analogously we show that $Int_{1\wedge 2}|p| \cup \{x\} \in \Omega_2$. Hence $x \in \tau_{1\wedge 2}|p|$.

Let us show that the induction rule is valid in the class of all T_D-intersection closed bi-topological spaces. The proof goes by contraposition. Assume not $\vdash p \to C_Bq$. This means that for some T_D-intersection closed, bi-topological model $\mathcal{M} = (X, \Omega_1, \Omega_2, V)$ and a point $x \in X$ it holds that: $x \Vdash p$ while $x \nVdash C_Bq$. We want to show that not $\vdash p \to \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$. It suffices to find a T_D-intersection closed bi-topological model which falsifies the formula. For such a model one could take $\mathcal{M}' = (X, \Omega_1 \cap \Omega_2, \Omega_1 \cap \Omega_2, V)$. Indeed as $(X, \Omega_1, \Omega_2, V)$ is T_D-intersection closed, the topology $\Omega_1 \cap \Omega_2$ satisfies the T_D-separation axiom. Besides since in $\mathcal{M}'$ both topologies are the same, their intersection is also $\Omega_1 \cap \Omega_2$ and hence again is a T_D-space. Now it is immediate that $\mathcal{M}', x \nVdash p \to \Box_1(p \wedge q) \wedge \Box_2(p \wedge q)$. This is because by construction of $\mathcal{M}'$ we have $\mathcal{M}', x \nVdash \Box_i q$ iff $\mathcal{M}, x \nVdash C_{\scriptscriptstyle B} q$ for every $x \in X$ and $i \in \{1, 2\}$.

(Completeness) Assume $\mathbf{K4_2^C} \nvdash \varphi$. According to Theorem 1 there exist a tree model $M^t = (W^t, R_1^t, R_2^t, V)$ which falsifies φ. We know that $(R_1 \cup R_2)^+$ is irreflexive and transitive order (see Note 1). By applying Proposition 3 we get that the formula φ is falsified in the corresponding bi-topological model $(W^t, \Omega_{R_1^t}, \Omega_{R_2^t}, V)$, which is T_D-intersection closed because of Fact 5, Proposition 2 and Note 1.

We can now show how the semantical definition of common belief $C_{\scriptscriptstyle B}\varphi$ as a colimit of the intersection topology meshes with the general equilibrium concept: on topological models the two operators $C_{\scriptscriptstyle B}$ and C_ν coincide.

Theorem 9. *For every bi-topological model $\mathcal{M} = (X, \Omega_1, \Omega_2, V)$ and an arbitrary formula φ the following equality holds: $\nu.p(\tau_1(|\varphi|) \cap \tau_2(|\varphi|) \cap \tau_1(p) \cap \tau_2(p)) = \tau_{1\wedge 2}(|\varphi|)$.*

Proof. That $\tau_{1\wedge 2}(|\varphi|)$ is a fixpoint of the operator $F(p) = \tau_1(|\varphi|) \cap \tau_2(|\varphi|) \cap \tau_1(p) \cap \tau_2(p)$ follows from the soundness proof of the equilibrium axiom, see Theorem 8. Now let us show that $\tau_{1\wedge 2}(|\varphi|)$ is the greatest fixpoint of $F(p)$. Take an arbitrary fixpoint B of the operator $F(p)$. That B is a fixpoint immediately implies that $B \subseteq \tau_1(|\varphi|) \cap \tau_2(|\varphi|) \cap \tau_1(B) \cap \tau_2(B)$. By 1) of Fact 4 we have $B \subseteq Int_i(B) = \tau_i(B) \cap B$ for each $i \in \{1, 2\}$. Hence $B = Int_{1\wedge 2}(B)$ where $Int_{1\wedge 2}$ is the interior operator in the intersection topology of the two topologies. Now let us show that for every $x \in B$ the set $\{x\} \cup (B \cap |\varphi|)$ is open in the intersection of the two topologies. Take an arbitrary point $y \in \{x\} \cup (B \cap |\varphi|)$. Since $y \in B \subseteq \tau_1(|\varphi|)$ we know that there exists an open neighborhood $U_y^1 \in \Omega_1$ of y such that $U_y^1 - \{y\} \subseteq |\varphi|$. This means that $B \cap U_y^1 \in \Omega_1$ and $B \cap U_y^1 \subseteq \{x\} \cup (B \cap |\varphi|)$. This means that for every point $y \in \{x\} \cup (B \cap |\varphi|)$ there is an open neighborhood $B \cap U_y^1 \in \Omega_1$ of y such that $B \cap U_y^1 \subseteq \{x\} \cup (B \cap |\varphi|)$ hence $\{x\} \cup (B \cap |\varphi|) \in \Omega_1$. In exactly the same way we show that $\{x\} \cup (B \cap |\varphi|) \in \Omega_2$. Hence $\{x\} \cup (B \cap |\varphi|) \in \Omega_1 \cap \Omega_2$. This means that $x \in \tau_{1\wedge 2}(|\varphi|)$ since there exists an open neighborhood $U_{1\wedge 2} = \{x\} \cup (B \cap |\varphi|) \in \Omega_1 \cap \Omega_2$ with $U_{1\wedge 2} - \{x\} \in |\varphi|$.

4 From Belief to Knowledge

In this section we discuss the connection between the logics of common knowledge $\mathbf{S4_2^C}$ and common belief $\mathbf{K4_2^C}$. This connection generalizes the existing splitting translation between $\mathbf{S4}$-logics and $\mathbf{K4}$-logics. As a result we obtain a validity preserving translation from $\mathbf{S4_2^C}$ formulas to $\mathbf{K4_2^C}$ formulas in which common knowledge is expressed in terms of common belief.

Definition 10. *The normal modal logic $\mathbf{S4_2^C}$ is defined in a modal language with infinite set of propositional letters $p, q, r..$ and connectives $\vee, \wedge, \neg, \Box_1, \Box_2, C_{\scriptscriptstyle K}$, where the formulas are constructed in a standard way.*

• *The axioms are all classical tautologies, each box satisfies all $S4$ axioms and in addition we have equilibrium axiom for common knowledge operator:*

$$(equi) : C_{_K} p \leftrightarrow p \wedge \Box_1 C_{_K} p \wedge \Box_2 C_{_K} p$$

• *The rules of inference are: Modus-ponens, Substitution, Necessitation for $\Box_1$ and $\Box_2$ and the induction rule:*

$$(ind) : \frac{\vdash \varphi \to \Box_1(\varphi \wedge \psi) \wedge \Box_2(\varphi \wedge \psi)}{\vdash \varphi \to C_{_K} \psi}$$

for an arbitrary formulas φ and ψ of the language.

The Kripke semantics for the modal logic $\mathbf{S4_2^C}$ is provided by the reflexive and transitive, bi-relational Kripke frames. For interpreting the common knowledge operator $C_{_K}$, the reflexive, transitive closure of a union relation is used.

Definition 11. *The reflexive, transitive closure $R^\star$ of a relation $R \subseteq W \times W$ is defined in the following way: $R^\star = R^+ \cup \{(w, w)|w \in W\}$.*

The satisfaction of formulas is definition in the following way.

Definition 12. *For a given bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ the satisfaction of a formula at a point $w \in W$ is defined inductively as follows:*
$w \Vdash p$ *iff* $w \in V(p)$,
$w \Vdash \alpha \wedge \beta$ *iff* $w \Vdash \alpha$ *and* $w \Vdash \beta$,
$w \Vdash \neg\alpha$ *iff* $w \nVdash \alpha$,
$w \Vdash \Box_i\varphi$ *iff* $(\forall v)(wR_iv \Rightarrow v \Vdash \varphi)$,
$w \Vdash C_{_K}\varphi$ *iff* $(\forall v)(w(R_1 \cup R_2)^\star v \Rightarrow v \Vdash \varphi)$.

Fact 10 *[16] The modal logic $\mathbf{S4_2^C}$ is sound and complete with respect to the class of all finite, reflexive, bi-transitive Kripke frames.*

Definition 13. *Consider the following function from the set of formulas in $\mathbf{S4_2^C}$ to the set of formulas in $\mathbf{K4_2^C}$.*
$Sp(p) = p$ *for every propositional letter* p,
$Sp(\neg\alpha \vee \beta) = \neg Sp(\alpha) \vee Sp(\beta)$,
$Sp(\Box_i\alpha) = \Box_i Sp(\alpha) \wedge Sp(\alpha)$,
$Sp(C_{_K}\alpha) = C_{_B} Sp(\alpha) \wedge Sp(\alpha)$.

Theorem 11. $\vdash_{\mathbf{S4_2^C}} \varphi$ *iff* $\vdash_{\mathbf{K4_2^C}} Sp(\varphi)$.

Proof. We prove the theorem by a semantical argument using the Kripke completeness results, see Proposition 1 and Fact 10. Let us first show by induction on the length of formula that for every bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ and every $w \in W$ the following holds:

$$(a) \quad \mathcal{M}^\star = (W, R_1^\star, R_2^\star, V), w \Vdash \varphi \text{ iff } \mathcal{M}^+ = (W, R_1^+, R_2^+, V), w \Vdash Sp(\varphi).$$

The only nonstandard case is when $\varphi = C_K \psi$. Assume $\mathcal{M}^\star, w \Vdash C_K \psi$. By the definition of $(R_1 \cup R_2)^\star$ this means that $\mathcal{M}^\star, w \Vdash \psi$ and for every w' such that $w(R_1 \cup R_2)^\star w'$, we have $\mathcal{M}^\star, w' \Vdash \psi$. Now by the induction hypotheses we have that $\mathcal{M}^+, w \Vdash \psi$ and $\mathcal{M}^+, w' \Vdash \psi$. Since w' was arbitrary $(R_1 \cup R_2)^\star$ successor of w we have $\mathcal{M}^+, w \Vdash C_B \psi$. This is because $(R_1 \cup R_2)^\star \supseteq (R_1 \cup R_2)^+$. Hence we get $\mathcal{M}^+, w \Vdash C_B \psi \wedge \psi$. The converse direction follows by the same argument.

Now assume $\vdash_{\mathbf{S4_2^C}} \varphi$. By fact 10 this means that φ is valid in every reflexive and transitive, bi-relational model. Take arbitrary transitive, bi-relational model $\mathcal{M}$. Then by assumption we have $\mathcal{M}^\star \Vdash \varphi$. Hence by (a) we have that $\mathcal{M} \Vdash Sp(\varphi)$. As $\mathcal{M}$ was arbitrary transitive, bi-relational model from Proposition 1 we get that $\vdash_{\mathbf{K4_2^C}} Sp(\varphi)$. Conversely assume $\vdash_{\mathbf{K4_2^C}} Sp(\varphi)$. Then by Proposition 1, $Sp(\varphi)$ is valid in the class of all transitive, bi-relational models. Take arbitrary reflexive and transitive, bi-relational model $\mathcal{N}$. Then $\mathcal{N} \Vdash Sp(\varphi)$ because $\mathcal{N} = \mathcal{N}^+$. So by (a) we have that $\mathcal{N}^\star \Vdash \varphi$. Now as $\mathcal{N}$ was reflexive and transitive, $\mathcal{N}^\star = \mathcal{N}$, hence $\mathcal{N} \Vdash \varphi$. Sinse $\mathcal{N}$ was arbitrary reflexive and transitive, bi-relational model, by Fact 10 we have $\vdash_{\mathbf{S4_2^C}} \varphi$.

5 Conclusions

Our main aim in this paper has been to extend the work of [13] on topological semantics for common knowledge by interpreting a common belief operator on the intersection of two topologies in a bi-topological model. In particular we considered a logic $\mathbf{K4_2^C}$ of common belief for normal agents, first under a Kripke, relational semantics, showing it to have the finite model property and the tree model property. We then showed that $\mathbf{K4_2^C}$ is the modal logic of all T_D-intersection closed, bi-topological spaces with a derived set interpretation of modalities and we saw how the common knowledge logic $\mathbf{S4_2^C}$ can be embedded in $\mathbf{K4_2^C}$ via the splitting translation that maps $C_K p$ into $p \wedge C_B p$.

While preparing the final draft of this work, we came across the article [27] by Lismont and Mongin. This paper treats several logics of common belief including one that is equivalent to $\mathbf{K4_2^C}$. Besides a relational semantics, the authors also consider a more general neighborhood semantics and discuss the equilibrium conception of common belief in this setting. While the semantics and methods of [27] are formally different to ours, there are obvious similarities. Though it is beyond the scope of this paper, a detailed comparison of our topological approach with the neighborhood systems of [27] would be a worthwhile exercise for the future. Another direction for future work is to look for concrete topological structures which would fully capture the behavior of the logic $\mathbf{K4_2^C}$ or some of it's extensions.

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