On One Nonlinear Mathematical Model of Blood Circulation with the Vessel Walls Reaction within the Hereditary Theory

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Abstract

The research demonstrates sufficient conditions of the existence and uniqueness for the solution in the oscillation mathematical model of the blood flow under nonlinear dissipative forces action within the theory of hereditary tube with biofactor. These results facilitated the application of different (explicit and implicit) numerical methods in further studies of the dynamical characteristics of solutions in the considered oscillation mathematical models. Numerical integration of the movement equations by Runge-Kutta 4th order method and Geer 2nd order method in a model case within this research enabled the estimation of the influence of different physical and mechanical factors on the amplitude and frequency of the oscillation process. The use of hybrid methods for the oscillation modeling in the nonlinear isotropic elastic environment on the example of a vessel enabled the formulation of the equation of an object's mechanical state based on energy approaches and the theory of mechanical fields in the continuous environments.

Keywords 1

Mathematical Model, Nonlinear Vibrations, Biofactor, Blood circulation, Vessel.

1. Introduction

Modern social development trends such as the problems of human survival and healthy lifestyle preservation are closely interconnected with the general human problems. Under such circumstances, top priority tasks are to improve the quality of human life, to devise a formula for active longevity, and to raise the individual living standards. Physical and spiritual human self-awareness should be also considered. That is why the problem of an adequate mathematical modeling of the processes in living organisms is the problem of current interest for the modern healthcare and science in general.

The spectrum of the considered problems is so wide that the whole investigation review is almost impossible. Brief but sufficiently capacious review of the mathematical models for many medical-biological processes, based on the well-known models and methods of the mechanics of continuous media is presented in [1]. More deepened analysis of medical-biological and mathematical aspects is proposed, particularly, in [2-10].

Numerical modeling of the biomechanical processes in the medical practice is realized using the models of mechanics of continuous media and numerical methods of solving the corresponding partial differential equations systems. Such modeling takes into account the development and realization of the numerical methods, adapted to the specified concrete tasks, development of the numerical method algorithm and its program package, visualization of the obtained results. To study some medical processes there is necessary to solve numerically the differential equations systems [7]. Biological and medical problems involving to numerical solutions of the partial differential equations are described

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in the papers [6-10]. Rheological relationships for the biological continuous media are developed in [11]. The mechanical model of the heart was considered in [12, 13]. Description of the simplest mathematical models of the circulatory system and heart one can find in [14-19]. Circulatory system consisting from the large and lesser circulation, possesses very serious and different functions, that is why their modeling in the normal and pathological conditions, is very important task in medicine. For today the most adequacy to the real physical circulatory systems there are dynamical models of the pulsating flows of the incompressible fluids in the elastic tubes system.

2. Hemodynamics equation in the linear mathematical models of blood circulation

Hemodynamics equation differs in every case [20].

1. The linear elastic tube. Oscillations equation of the blood flow is the next:

$$\frac{\partial}{\partial x}\left(\frac{1}{R^3}\frac{\partial Q}{\partial x}\right)-\frac{2\rho}{R^2}\left(\frac{\partial^2 Q}{\partial t^2}+\frac{8\chi}{R^2}\frac{\partial Q}{\partial t}\right)=0,$$

where $Q = SU$ is blood consumption, $S(x)$ is area of the tube cross-section, $R(x)$ is the tube radius, $U(x,t)$ is blood averaged flow rate, ρ is blood density, χ is coefficient of kinematic viscosity.

2. The linear hereditary tube. Oscillations equation is:

$$\sum_{s=1}^{n}\delta_s E_s\frac{\partial}{\partial x}\left[\frac{1}{R^3}\left(\frac{\partial Q}{\partial x}-\int_0^{\infty}\Gamma_s(\theta)\cdot\frac{\partial Q(x,t-\theta)}{\partial x}d\theta\right)\right]-\frac{2\rho}{R^2}\left(\frac{\partial^2 Q}{\partial t^2}+\frac{8\chi}{R^2}\frac{\partial Q}{\partial t}\right)=0,$$

where $\sigma_s = e_s E_s$, σ_s is stress, e_s is deformation in the corresponding s is layer of a multilayer cylindrical tube (vessel); $\Gamma_s(t)$ is some hereditary function.

3. The linear hereditary biofactor tube. Oscillations equation is

$$\sum_{s=1}^{n}\delta_s E_s\frac{\partial}{\partial x}\left[\frac{1}{R^3}\left((1-A_s)\frac{\partial Q}{\partial x}+A_s\tau\frac{\partial^2 Q}{\partial x\partial t}\right)\right]-\frac{2\rho}{R^2}\left(\frac{\partial^2 Q}{\partial t^2}+\frac{8\chi}{R^2}\frac{\partial Q}{\partial t}\right)=0,$$

where $0 < A_s < 1$, τ is time delay of the reaction, $\tau \ll t$, $\sigma_s = (1-\sigma_s)\sigma_s^0 + A_s\tau\frac{\partial\sigma_s^0}{\partial\tau}$, $\sigma_s^0 = E_s\frac{W}{R}$, W is radial displacement of the wall for the multilayer package as a whole.

Solutions of these equations can be found as finite sum of the main oscillation and the higher harmonics, using the harmonic analysis methods.

3. Problem statement. Mathematical models of blood circulation within the nonlinear theory

Study of the wave propagation process in the deformed tubes with the liquid leaking through the tube is widespread applicated [21, 22]. These problems are actual, in particular, in case of blood circulation modeling in alive organisms. Problems of blood flows and oscillations propagation in large blood vessels are very important to understand the functioning, regulation and control the cardiovascular system [23]. As follows, diagnostics, surgery and prosthetics are bound up the hemodynamics [20, 24]. In the mathematical modeling of blood flow there is considered pulsed systaltic blood flow in the multilayer elastic or viscous elastic tube with the variable cross-section. More complicated mathematical models of blood circulation in the tubes that possess reaction on the external action (biofactor). This type models describe blood circulation in arteries and veins.

To study complicated impulses, specific to the circulatory system, it is often necessary to consider the nonlinear models instead the linear ones. It is impossible to find the exact solution in this case. Background of the nonlinear mathematical models study is numerical (computer) modeling. Application of this approach also can't be universal study method due to the other problems, for example, procedure convergence, numerical method's stability, accuracy of computation. That's why it is reasonable to develop hybrid methods to study the nonlinear oscillations mathematical models combining both the qualitative and numerical approaches [25, 26]. It is realized thorough qualitative description of the solution's characteristics respectively to the problem. Usually, it is based on Galerkin method or on its different modifications. After that the numerical methods are applicated to find the approximate solutions. Herewith choice of the numerical method is of no principle from the theoretical aspect, and can be determined by effectiveness of the numerical realization only.

Using all mentioned above, let's consider the oscillations equation of blood flow in the nonlinear isotropic environment. It would be studied the problem of the blood flow oscillations equation within the mathematical model of the linear hereditary tube with biofactors and nonlinear external dissipative forces in the form

$$\frac{\partial^2 Q}{\partial t^2} = a^2 \frac{\partial^2 Q}{\partial x^2} + \frac{\zeta}{\rho_l} \frac{\partial^3 Q}{\partial x^2 \partial t} - \frac{\nu}{\rho_l} \left|\frac{\partial Q}{\partial t}\right|^{p-2} \frac{\partial Q}{\partial t}, \tag{1}$$

where $a = \sqrt{\frac{N_l}{\rho_l}}$, ρ_l is the linear blood density in the finite tube with fixed length l, N_l is uniformly distributed by tube length force, causing the initial oscillations, ν, ζ are the coefficients of the external and internal dissipations of the environment respectively, $p > 2$. The nonlinear oscillations of the tube with the constant radius of the cross-section R in the case of the initial amplitude replacement and the initial zero rate would be considered. The case of rigidly fixed by length tube also would be considered. It is reasonable to study the equation (1) in the rectangle $\Pi_T = [0; l] \times [0; T]$ mixed problem with the initial conditions

$$Q(x,0) = Q_0(x), \qquad \frac{\partial Q(x,0)}{\partial t} = 0 \tag{2}$$

and the boundary conditions

$$Q(0,t) = Q(l,t) = 0. \tag{3}$$

For the case $p = 2$ the equation (1) is considered in the previous chapter within the linear elastic tube with the reaction (biofactor) model. In this paper the linear model is involved by the nonlinear factors.

4. Model case of the elastic tube with the reaction and computer modeling results

For the mathematical modeling of the free nonlinear small transversal oscillations there is used long fixed on the endpoints tube under the force action on the unit of length N_l. Free oscillations of the blood flow are described by the problem (1), (2), (3). Under these conditions the space discretization of the equation (1) is realized. Let n be the quantity of the discretization components, and Δx be difference interval over the coordinate x. Solving of the problem (1), (2), (3) results in the numerical integration on the time interval of some difference equations system under some initial conditions. Computer modeling of the transient processes was realized on the model example of the analysis of the small transversal oscillations of the elastic thin tube $R = 0{,}003\ m$ with the finite length. Tube length is 1 m, blood density $\rho = 1055 \frac{\text{kg}}{m^3}$, wall thickness is $h\ m$. Tube is under the initial pertubation of the force applied to its center in the dilation direction (that means perpendicular to the line length). System parameters are the next: $N_l = 50 \frac{N}{m}$, $\Delta x = 0{,}1\ m, \zeta = 1 \frac{\text{kg}\cdot m}{s}$. Integration of the equations in the mechanical mode is realized via the explicit Runge-Kutta fourth order method and implicit second order Geer method. Numerical results almost coinside. Integration step of the explicit Runge-Kutta method is $1 \cdot 10^{-5} s$, implicit Geer method is $1 \cdot 10^{-4} s$. The nonlinear algebraic equations system on every step by the variable t is solved via the simple iteration method. Three modes of the object were studied. The first mode presents the oscillations of the sufficiently small flows in the tube (the linear internal dissipation of the mechanical energy is present only, $\nu = 0$). The second mode presents half-filled tube (the linear internal and linear external dissipations of the mechanical energy are present, $p = 2$, $\nu = 3$). The third mode presents the oscillations of the filled tube (the linear internal and the nonlinear external dissipations of the mechanical energy are present, $p = 4{,}3$, $\nu = 3$). Obviously, that these assumptions are adapted, but even in this case with the sufficient adequacy extent there are described the real physical processes in the object. Thus, there realized three experiments taking into consideration mentioned modes. To confirm the validity of the system model also were carried out two additional experiments consisting in study of the transitional processes with different tube thickness. The first experiment examined $h = 0{,}002$ *m* (Fig. 1, Fig. 2), the second experiment examined $h = 0{,}001$ *m* (Fig. 3).

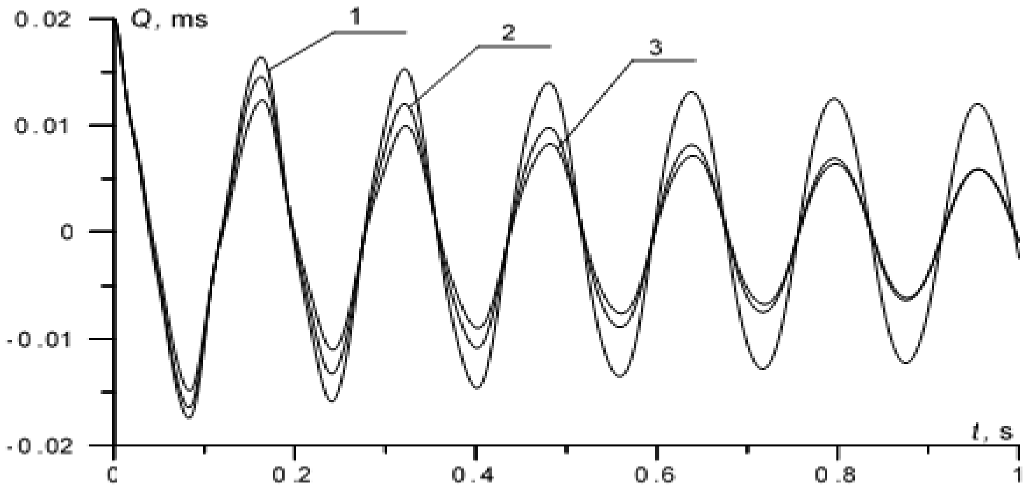

Figure 1: Transient motions of the tube central component $t \in [0;1]$, ($h = 0{,}002m$): 1 is the first experiment, 2 is the second experiment, 3 is the third experiment

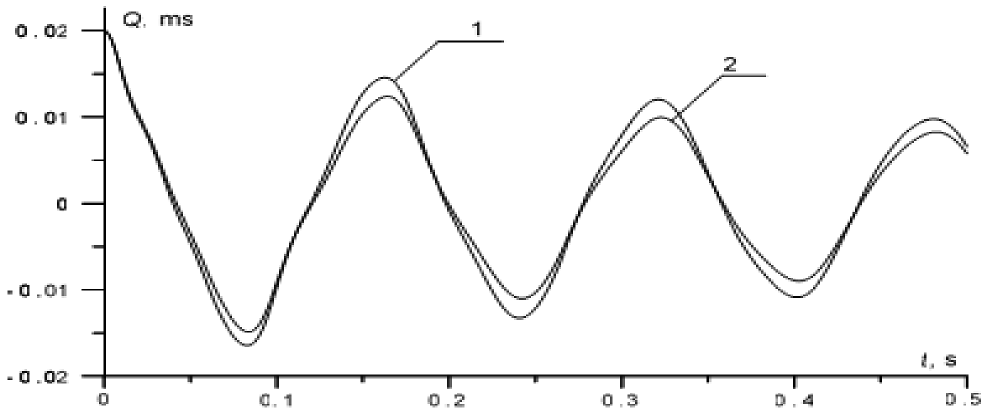

Figure 2: Transient motions of the tube central component ($h = 0{,}002m$): 1 is the second experiment, 2 is the third experiment at time span $t \in [0;0{,}5]$

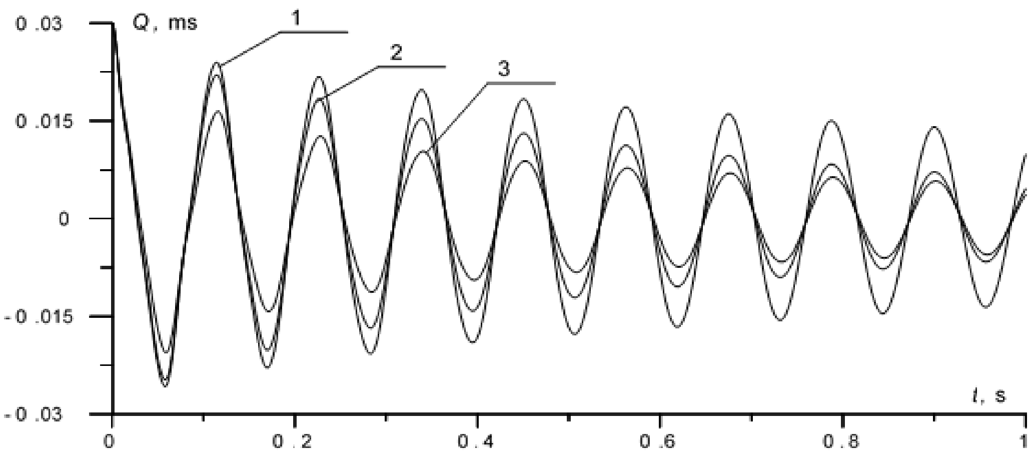

Figure 3: Transient motions of the tube central component ($h = 0{,}001m$): 1 is the first experiment, 2 is the second experiment, 3 is the third experiment

Analyzing the family of curves, one can see the essential influence of the internal dissipative processes on the tube oscillations. Partly filled tube can be treated as an object with the linear external dissipation in contradistinction to the fully filled tube. Turbulent processes of the blood circulation cause the nonlinear influence on the vessel walls. This influence depends on the vessel wall thickness. This fact is clearly fixed on the Fig. 3. Reducing of the tube thickness causes the increasing of the eigenoscillation frequency and contrariwise. This fact absolutely corresponds to the classical elasticity theory. The oscillations damping in the tube with the less thickness are more intensive being dependent on the internal processes in the vessel body. The nonlinear characteristics of the blood flow are manifested stronger in the vessels with the thin walls, that is well-understood from the physical point of view.

5. Conclusions

The elaboration of the mathematical models of the physiological processes in the able-bodied organism, and also medical problems that follow in the sick mode of the patient, can be considered as mathematical modeling domain that is intensively developed. Hereby the qualitative study of the mathematical model and the next numerical modeling often are effective and available instrument of the biological and medical problems investigation. To confirm the problem correctness in the nonlinear mathematical model of the blood circulation in the paper are used the fundamental methods of the nonlinear boundary problems general theory. Basing on the results of the numerical modeling there is proved the sufficient adequacy of the obtained model to the real prototype. It is shown that the nonlinear medium promotes to the quicker oscillations damping and causes the inharmonic processes in the system. The well-known fact, that increasing of the vessel walls thickness causes the reducing eigenoscillations frequency of the system and contrariwise also is reaffirmed.

6. References

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