Undecidability of Higher-Order Unification Formalised in Coq

Simon Spies, Yannick Forster

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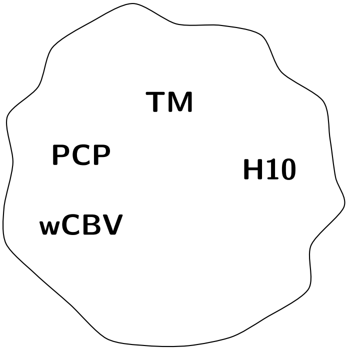
CPP'20

SIC Saarland Informatics Campus
SAARLAND UNIVERSITY
COMPUTER SCIENCE
UNIVERSITY OF CAMBRIDGE

Extending the Coq Library of Undecidable Problems



Library of Undecidable Problems[1]

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Higher-Order Unification

[1] For an overview of the library see the talk at 4pm on Saturday at CoqPL

Higher-Order Unification in Action

Emacs-x86_64-10_14 Prelude - ~/Projects/playground/natind.v

```
Lemma plus_zero : ∀ n, n + 0 = n.
Proof.
  Check nat_ind.
  apply nat_ind with (P := λ n ⇒ n + 0 = n).
  reflexivity.
  cbn; now intros ? ->.
Qed.
```

```
-:---  natind.v      29% of 571  (11,0)     (Coq Script(1-) +2 company-    yas hs Holes Outl FlyC company i
```

```
U:%%-  *response*    All of 0    (1,0)      (Coq Response +2 FlyC company ivy EditorConfig Projectile supe
```

Higher-Order Unification in Action

Emacs-x86_64-10_14 Prelude - ~/Projects/playground/natind.v

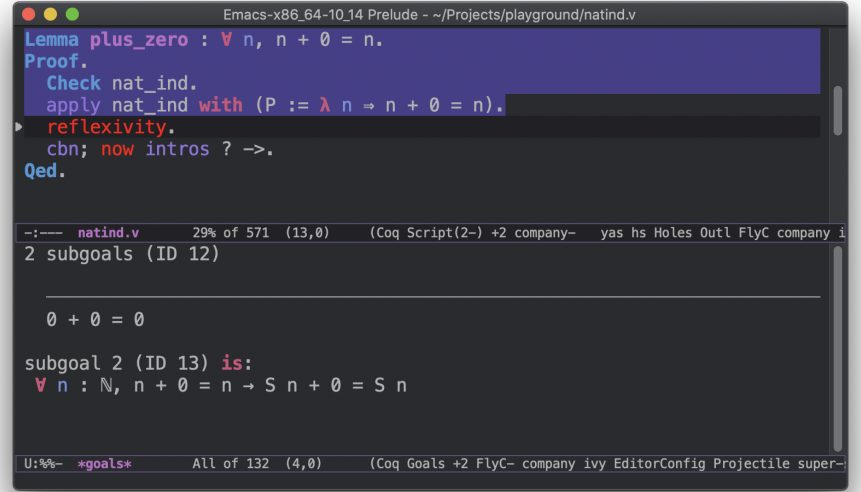
```
Lemma plus_zero : ∀ n, n + 0 = n.
Proof.
  Check nat_ind.
  apply nat_ind with (P := λ n ⇒ n + 0 = n).
  reflexivity.
  cbn; now intros ? ->.
Qed.
```

```
-:---  natind.v      29% of 571  (12,0)     (Coq Script(1-) +2 company-    yas hs Holes Outl FlyC company i
```

```
nat_ind
     : ∀ P : ℕ → ℙ, P 0 → (∀ n : ℕ, P n → P (S n)) → ∀ n : ℕ, P n
```

```
U:%%-  *response*    All of 102  (2,65)     (Coq Response +2 FlyC company ivy EditorConfig Projectile supe
```

Higher-Order Unification in Action



Higher-Order Unification in Action

```
Emacs-x86_64-10_14 Prelude - ~/Projects/playground/natind.v
Lemma plus_zero : ∀ n, n + 0 = n.
Proof.
  Check nat_ind.
  apply nat_ind.
  reflexivity.
  cbn; now intros ? ->.
Qed.

-:---  natind.v       30% of 540  (13,0)     (Coq Script(2-) +2 company-   yas hs Holes Outl FlyC company i
2 subgoals (ID 9)

  ____________________________________________________________
  0 + 0 = 0

subgoal 2 (ID 10) is:
 ∀ n : ℕ, n + 0 = n → S n + 0 = S n

U:%%-  *goals*        All of 131  (4,0)      (Coq Goals +2 FlyC- company ivy EditorConfig Projectile super-
Wrote /Users/spies/Projects/playground/natind.v
```

Higher-Order Unification in Theory

Instance

$$P : \mathbb{N} \to \mathbb{P} \vdash (\forall n.\ P\ n) \stackrel{?}{=} (\forall n.\ n + 0 = n) : \mathbb{P}$$

containing constants, bound variables, free variables, and types.

Higher-Order Unification in Theory

Instance

$$P : \mathbb{N} \to \mathbb{P} \vdash (\forall n.\ P\ n) \overset{?}{=} (\forall n.\ n + 0 = n) : \mathbb{P}$$

containing constants, bound variables, free variables, and types.

Solution The substitution

$$(\lambda n. n + 0 = n)/P$$

since

$$(\forall n.\ (\lambda n. n + 0 = n)\ n) \equiv_\beta (\forall n.\ n + 0 = n)$$

Why is this hard?

Emacs-x86_64-10_14 Prelude - ~/Projects/playground/natind.v

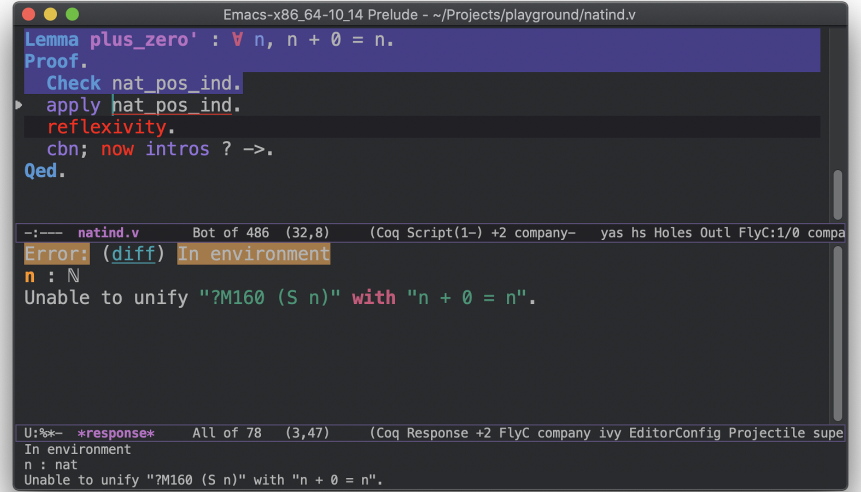
```
Lemma plus_zero' : ∀ n, n + 0 = n.
Proof.
  Check nat_pos_ind.
  apply nat_pos_ind.
  reflexivity.
  cbn; now intros ? ->.
Qed.
```

```
-:---  natind.v       Bot of 486   (32,0)      (Coq Script(1-) +2 company-    yas hs Holes Outl FlyC:1/0 compa
```

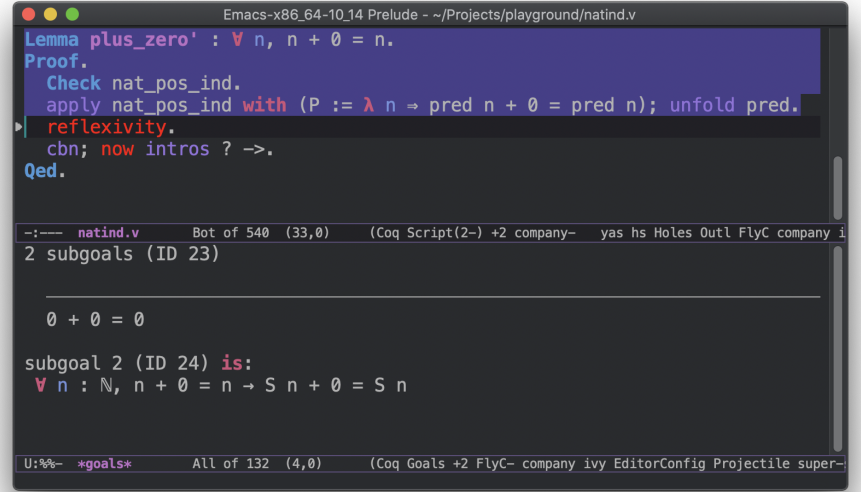
```
nat_pos_ind
     : ∀ P : ℕ → ℙ,
       P 1 → (∀ n : ℕ, P (S n) → P (S (S n))) → ∀ n : ℕ, P (S n)
```

```
U:%%-  *response*     All of 125   (3,64)      (Coq Response +2 FlyC company ivy EditorConfig Projectile supe
```

Why is this hard?



Why is this hard?



Overview

Overview

Huet (1973)

$$\mathbf{PCP} \preceq \mathbf{U}_{3+k}$$

PCP Post-correspondence problem

Overview

Huet (1973)

$\textsf{PCP} \preceq \textsf{U}_{3+k}$

Goldfarb (1981)

$\textsf{H10} \preceq \textsf{U}_2$

H10 Hilbert's tenth problem **PCP** Post-correspondence problem

Overview

Huet (1973)

$\mathsf{PCP} \preceq \mathsf{U}_{3+k}$

Goldfarb (1981)

$\mathsf{H10} \preceq \mathsf{U}_2$

Dowek (2001)

$\mathsf{H10} \preceq \mathsf{U}$

H10 Hilbert's tenth problem

PCP Post-correspondence problem

Overview

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Dowek (2001)

$\mathsf{H10} \preceq \mathsf{U}$

Our Contributions

1. Coq formalisation
2. Simplification of Goldfarb's proof
3. Simplification of Huet's proof
4. Same Calculus

H10 Hilbert's tenth problem **PCP** Post-correspondence problem

Simply-Typed λ-Calculus

$$s, t ::= x \mid c \mid \lambda x.s \mid s\ t \qquad (c : \mathcal{C})$$
$$A, B ::= \alpha \mid A \to B$$
$$\Gamma, \Delta ::= x_1 : A_1, \ldots, x_n : A_n$$

Equality:	β-equivalence	$s \equiv_\beta t$
Substitution:	capture-avoiding	$s[\sigma]$
Typing:	Curry-style	$\Gamma \vdash s : A$

Higher-Order Unification

$$\mathbf{U}\ (\Gamma \vdash s \stackrel{?}{=} t : A)$$

Higher-Order Unification

$$\mathbf{U}\ (\Gamma \vdash s \stackrel{?}{=} t : A) :=$$

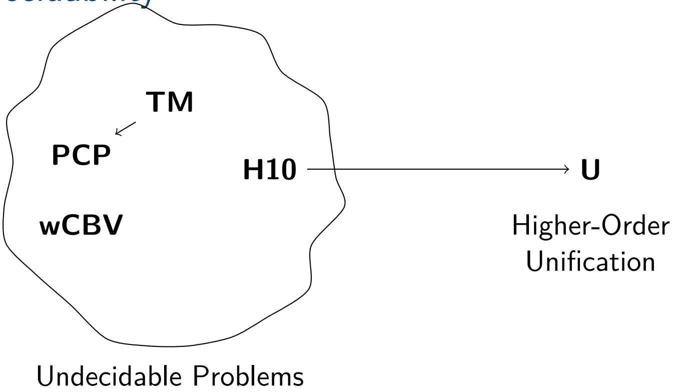
$$\exists \sigma \qquad s[\sigma] \equiv_\beta t[\sigma]$$

Higher-Order Unification

$$\mathbf{U}\ (\Gamma \vdash s \stackrel{?}{=} t : A) :=$$
$$\exists \sigma \Delta.\ \Delta \vdash \sigma : \Gamma \quad \text{and} \quad s[\sigma] \equiv_\beta t[\sigma]$$

where $\Delta \vdash \sigma : \Gamma := \forall (x : A) \in \Gamma.\ \Delta \vdash \sigma x : A$

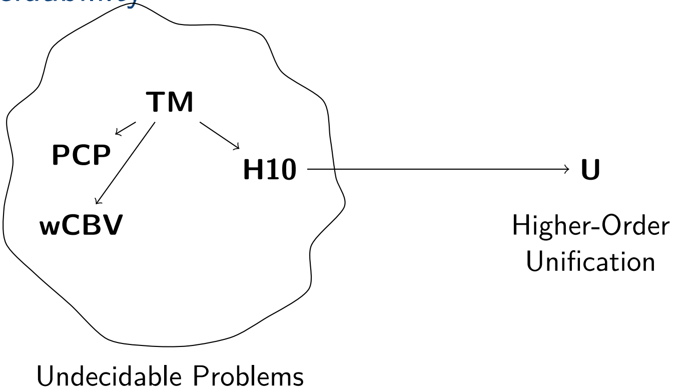
Traditional Undecidability



P undec. iff there is no TM deciding **P**

P $\preceq$ **Q** iff there is a TM computable function f such that $\forall x.$ **P**(x) iff **Q**$(f(x))$

Synthetic Undecidabililty



P undec.	iff	$\mathbf{TM} \preceq \mathbf{P}$
$\mathbf{P} \preceq \mathbf{Q}$	iff	there is a Coq function f such that
		$\forall x.\ \mathbf{P}(x)$ iff $\mathbf{Q}(f(x))$

Reduction

$$\mathbf{H10} \quad \preceq \quad \mathbf{U}$$

Reduction

$$\textbf{H10} \preceq \textbf{SU} \preceq \textbf{U}$$

$$\textbf{SU}\ (\{\Gamma \vdash s_i \stackrel{?}{=} t_i : A_i \mid i = 1, \ldots, n\}) :=$$
$$\exists \sigma \Delta.\ \Delta \vdash \sigma : \Gamma \quad \text{and} \quad \forall i.\ s_i[\sigma] \equiv_\beta t_i[\sigma]$$

Hilbert's tenth problem

Diophantine Equations

$$
\begin{aligned}
d ::= &\; x \doteq 1 \\
\mid &\; x + y \doteq z \\
\mid &\; x \cdot y \doteq z
\end{aligned}
$$

$$
\begin{aligned}
\theta \vDash x \doteq 1 \quad &\text{iff} \quad \theta x = 1 \\
\theta \vDash x + y \doteq z \quad &\text{iff} \quad \theta x + \theta y = \theta z \\
\theta \vDash x \cdot y \doteq z \quad &\text{iff} \quad \theta x \cdot \theta y = \theta z
\end{aligned}
$$

Hilbert's tenth problem

$$\mathbf{H10}(D) := \exists\theta.\forall d \in D.\ \theta \vDash d$$

Diophantine Equations

$$d ::= x \doteq 1 \mid x + y \doteq z \mid x \cdot y \doteq z$$

$$\begin{aligned} \theta \vDash x \doteq 1 &\quad \text{iff} \quad \theta x = 1 \\ \theta \vDash x + y \doteq z &\quad \text{iff} \quad \theta x + \theta y = \theta z \\ \theta \vDash x \cdot y \doteq z &\quad \text{iff} \quad \theta x \cdot \theta y = \theta z \end{aligned}$$

H10 $\to$ **SU** following Dowek (2001)

$$\mathbf{H10} \preceq \mathbf{SU}$$

H10 → SU following Dowek (2001)

$$\textbf{H10}(D) \quad \text{iff} \quad \textbf{SU}(f(D))$$

H10 $\to$ SU following Dowek (2001)

$$\mathbf{H10}(D) \quad \text{iff} \quad \mathbf{SU}(f(D))$$

where f is given by

$$\begin{aligned} f(x \doteq 1) &:= x \stackrel{?}{=} [\![1]\!]_{\mathsf{cn}} \\ f(x + y \doteq z) &:= x \oplus y \stackrel{?}{=} z \\ f(x \cdot y \doteq z) &:= x \otimes y \stackrel{?}{=} z \end{aligned}$$

and for every variable x a characteristic equation CN x.

Church Numerals

$[\![n]\!]_{\mathsf{cn}} := \lambda a f. f^n\ a$ $\oplus$ faithful $\otimes$ faithful

Fragments

$$x \oplus y \stackrel{?}{=} z \qquad\qquad \lambda a f. x\ (y\ a\ f)\ f \stackrel{?}{=} z$$

where

$$x, y, z : \alpha \to (\alpha \to \alpha) \to \alpha$$

Fragments

Third-Order Unification

$$x \oplus y \stackrel{?}{=} z \qquad\qquad \lambda a f. x\ (y\ a\ f)\ f \stackrel{?}{=} z$$

where

$$x, y, z : \alpha \to (\alpha \to \alpha) \to \alpha$$

Fragments

First-Order Unification

$$\mathsf{g}\ u\ \mathsf{a} \stackrel{?}{=} \mathsf{g}\ \mathsf{a}\ v \qquad \mathsf{g}\ \mathsf{a}\ v \stackrel{?}{=} w \qquad \mathsf{g}\ u\ \mathsf{a} \stackrel{?}{=} u$$

Third-Order Unification

$$x \oplus y \stackrel{?}{=} z \qquad \lambda a f. x\ (y\ a\ f)\ f \stackrel{?}{=} z$$

where

$$\mathsf{g} : \alpha \to \alpha \to \alpha \qquad \mathsf{a} : \alpha \qquad x, y, z : \alpha \to (\alpha \to \alpha) \to \alpha \qquad u, v, w : \alpha$$

Fragments

First-Order Unification

$$\mathsf{g}\ u\ \mathsf{a} \stackrel{?}{=} \mathsf{g}\ \mathsf{a}\ v \qquad \mathsf{g}\ \mathsf{a}\ v \stackrel{?}{=} w \qquad \mathsf{g}\ u\ \mathsf{a} \stackrel{?}{=} u$$

Second-Order Unification

$$\mathsf{g}\ \mathsf{a} \stackrel{?}{=} h\ \mathsf{a} \qquad h\ \mathsf{a} \stackrel{?}{=} h\ (h\ \mathsf{a})$$

Third-Order Unification

$$x \oplus y \stackrel{?}{=} z \qquad \lambda a f. x\ (y\ a\ f)\ f \stackrel{?}{=} z$$

where

$$\mathsf{g} : \alpha \to \alpha \to \alpha \qquad \mathsf{a} : \alpha \qquad x, y, z : \alpha \to (\alpha \to \alpha) \to \alpha \qquad u, v, w : \alpha \qquad h : \alpha \to \alpha$$

Nth-Order Unification

$$\mathbf{U}_n\ (\Gamma \vdash_n s \overset{?}{=} t : A) :=$$
$$\exists \sigma \Delta.\ \Delta \vdash_n \sigma : \Gamma \quad \text{and} \quad s[\sigma] \equiv_\beta t[\sigma]$$

where $\Delta \vdash_n \sigma : \Gamma := \forall (x : A) \in \Gamma.\ \Delta \vdash_n \sigma x : A$

Conservativity

Conservativity

$$\mathbf{U}_n \preceq_{\mathsf{id}} \mathbf{U}_{n+k} \preceq_{\mathsf{id}} \mathbf{U} \quad \text{for } n \geq 1, k \geq 0$$

Corollary

$$\mathbf{U}_1 \preceq_{\mathsf{id}} \mathbf{U}_2 \preceq_{\mathsf{id}} \mathbf{U}_{2+k} \preceq_{\mathsf{id}} \mathbf{U}$$

Second-Order Undecidability following Goldfarb (1981)

$$\mathbf{H10} \preceq \mathbf{U}_2$$

with constants $\mathsf{g} : \alpha \to \alpha \to \alpha$ and $\mathsf{a} : \alpha$.

Goldfarb Numerals

$\boxed{[\![n]\!]_{\mathsf{cn}}}$

$$[\![n]\!]_{\mathsf{cn}} := \lambda a f.f^n\ a$$

to

$\boxed{[\![n]\!]_{\mathsf{gn}}}$

$$[\![n]\!]_{\mathsf{gn}} := \lambda a.\mathsf{S}^n\ a$$

where $\mathsf{S} := \mathsf{g}\ \mathsf{a}$ with $\mathsf{g} : \alpha \to \alpha \to \alpha$ and $\mathsf{a} : \alpha$

Goldfarb Numerals

$\boxed{[\![n]\!]_{\mathsf{cn}}}$

$$[\![n]\!]_{\mathsf{cn}} := \lambda a f. f^n\ a$$

to

$\boxed{[\![n]\!]_{\mathsf{gn}}}$

$$[\![n]\!]_{\mathsf{gn}} := \lambda a. \mathsf{S}^n\ a$$

where $\mathsf{S} := \mathsf{g}\ \mathsf{a}$ with $\mathsf{g} : \alpha \to \alpha \to \alpha$ and $\mathsf{a} : \alpha$

Operations

- ✓ addition
- ✓ characteristic equation
- ✗ multiplication

$$s \otimes t := \lambda a f. \underbrace{s\ a\ (\lambda b. t\ b\ f)}_{\text{3rd-order}}$$

Multiplication

Following Goldfarb (1981), the equation

$$x \cdot y \doteq z$$

is encoded as

$$\begin{aligned} & \lambda uv.G_{xyz}\ (\mathsf{g}\ (\mathsf{g}\ (z\ u)\ (x\ v))\ \mathsf{a})\ u\ v \\ \stackrel{?}{\doteq}\ & \lambda uv.\mathsf{g}\ (\mathsf{g}\ u\ v)\ (G_{xyz}\ \mathsf{a}\ (y\ u)\ (\mathsf{S}\ v)) \end{aligned}$$

where $G_{xyz} : \alpha \to \alpha \to \alpha \to \alpha$ and $x, y, z : \alpha \to \alpha$

Why? Explanation in the paper.

Contributions in the context of the library

$$\textbf{TM} \preceq \textbf{H10} \preceq \textbf{U}_2 \preceq \textbf{U}_{2+k} \preceq \textbf{U}$$

Library | Goldfarb | Conservativity

Recall

P undec. iff **TM** $\preceq$ **P**

P $\preceq$ **Q** iff there is a Coq function f such that $\forall x.\ \textbf{P}(x)$ iff $\textbf{Q}(f(x))$

Furthermore. . .

- First-Order Unification

  $\mathbf{U}_1$ is decidable

- Simplifying Huet (1973)

  $\mathbf{PCP} \preceq \mathbf{U}_3$ simplified to $\mathbf{MPCP} \preceq \mathbf{U}_3$

- Techniques for treating constants similar to Statman (1981)

$$\mathbf{U}_2^{\{\mathrm{g},\mathrm{a}\}} \preceq \mathbf{U}_2^{\{\mathrm{g}\}} \preceq \mathbf{U}_3^{\{\mathrm{g}\}} \preceq \mathbf{U}_3^{\emptyset}$$

- $\mathbf{U}, \mathbf{SU}, \mathbf{U}_n,$ and $\mathbf{SU}_n$ are enumerable

Future Work

- Decidability of monadic 2nd-order unification; Farmer (1988)
- Huet's unification procedure; Huet (1975)

Formalisation

Details

- De Bruijn indices
- Normalisation for the STLC
- Constant Replacement
- Meta Theory of the STLC

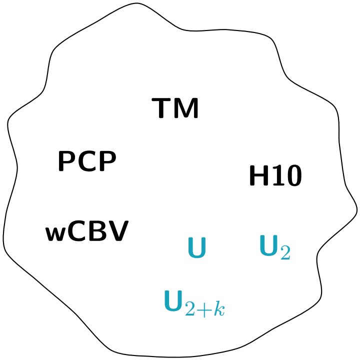
Coq Code

Unification	3000
Undecidability	450
Second-Order	1000
Other	3000
Total	7450

Tools

- *Autosubst 2* used for generating terms and substitution
- *Equations* used for defining first-order unification algorithm
- *Setoid Rewriting* used for reasoning about β-equivalence

Coq Library of Undecidable Problems



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???
Your Contribution

Library under uds-psl on Github

and *4pm on Saturday at CoqPL*

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1981. On the existence of closed terms in the typed λ calculus II: Transformations of unification problems. *Theoretical Computer Science*, 15(3):329–338.

Characteristic Equation

Iteration fulfills

$$f^n(fa) = f(f^n a)$$

We can show: Let s be normal.

$$\lambda af.s\ (f\ a)\ f \equiv_\beta \lambda af.f\ (s\ a\ f) \quad \text{iff} \quad s = [\![n]\!]_{\mathsf{cn}} \quad \text{for some } n : \mathbb{N}$$

where $[\![n]\!]_{\mathsf{cn}} := \lambda af.f^n\ a$.

Characteristic Equation

$$\mathsf{CN}\ x := \lambda af.x\ (f\ a)\ f \stackrel{?}{=} \lambda af.f\ (x\ a\ f)$$

SU $\preceq$ U

$$\textbf{SU}(E) \quad \text{iff} \quad \textbf{U}(f(E))$$

Proof.

Pick $f := \{\Gamma \vdash s_i \stackrel{?}{=} t_i : A_i \mid i = 1, \ldots, n\} \mapsto$

$$\Gamma \vdash \lambda h.h\ s_1 \cdots s_n \stackrel{?}{=} \lambda h.h\ t_1 \cdots t_n : A$$

where $A = (A_1 \to \cdots \to A_n \to \alpha) \to \alpha$. Follows with:

$$h\ u_1 \cdots u_n \equiv_\beta h\ v_1 \cdots v_n \ \text{ iff } \ \forall i.\ u_i \equiv_\beta v_i$$

Multiplication

Multiplication sequence

$$(0,0),(n,1),(2n,2),\cdots,(m\cdot n,m)$$

generated by

$$m\cdot n=p \quad \text{iff} \quad \exists X.\ (0,0) :: \mathsf{map\ step}\ X = X +\!\!+ [(p,m)]$$

where $\mathsf{step}(a,i) := (a+n, i+1)$.

Modified Post Correspondence Problem — **MPCP**

Given

$$\boxed{\frac{l_0}{r_0}} \quad \text{and} \quad \boxed{\frac{l_1}{r_1}} \cdots \boxed{\frac{l_n}{r_n}}$$

$$\textcircled{0} \qquad\qquad \textcircled{1} \qquad \textcircled{n}$$

Find Ordering

$$i_1, \ldots, i_k$$

Such that

$$l_0 l_{i_1} \cdots l_{i_k} = r_0 r_{i_1} \cdots r_{i_k}$$

Simplification of Huet's Proof

Original

$$\lambda u_1 u_0 h.h\ (f\ \overline{l_0} \cdots \overline{l_n})\ (f\ u_1 \cdots u_1) \stackrel{?}{=} \lambda u_1 u_0 h.h\ (f\ \overline{l_0} \cdots \overline{l_n})\ (u_1\ (d\ u_1))$$

where $f : (\alpha \to \alpha)^{n+1} \to \alpha \to \alpha$ and $d : (\alpha \to \alpha) \to \alpha$.

vs.

Simplification

$$\lambda u_1 u_0.\overline{l_0}\ (f\ \overline{l_0}\ \cdots\ \overline{l_n}) \stackrel{?}{=} \lambda u_1 u_0.\overline{r_0}\ (f\ \overline{r_0}\ \cdots\ \overline{r_n})$$

where $f : (\alpha \to \alpha)^{n+1} \to \alpha \to \alpha$.

Meta Theory of STLC

Small Challenges

$$\textit{If } h\ s \equiv_\beta h\ t,\ \textit{then } s \equiv_\beta t.$$

- *If* $s \succ s'$ *and* $\text{isLam}(\text{head } s')$ *then* $\text{isLam}(\text{head } s)$.
- *If* $s\ t \succ^* v$ *then* $s \succ^* s', t \succ^* t'$, *and* $v = s'\ t'$ *for some* s', t'
  or $s \succ^* \lambda x.s'$ *and* $\text{isLam}\ (\text{head } s)$ *for some* s'.
- *If* $s_1\ s_2 \equiv_\beta t_1\ t_2$, $\text{isVar}\ (\text{head } s_1)$, *and* $\text{isVar}\ (\text{head } t_1)$, *then* $s_1 \equiv_\beta t_1$ *and* $s_2 \equiv_\beta t_2$.

List Operations

$\boxed{S\ t}$	$\boxed{s\ T}$	$\boxed{\Lambda X.\ s}$
$\text{nil}\ t = t$	$s\ \text{nil} = s$	$\Lambda\text{nil}.\ s = s$
$(s :: S)\ t = s\ (S\ t)$	$s\ (t :: T) = (T\ s)\ t$	$\Lambda x :: X.\ s = \lambda x.\Lambda X.\ s$

Conservativity — $\mathbf{U}_n \subseteq \mathbf{U}$

Let $\Gamma \vdash_n s \overset{?}{=} t : A$.

$$s[\sigma] \equiv_\beta t[\sigma] \text{ for some } \Sigma \vdash_n \sigma : \Gamma$$
$$\text{iff}$$
$$s[\sigma] \equiv_\beta t[\sigma] \text{ for some } \Delta \vdash \sigma : \Gamma$$

Proof Sketch.
Replace free variables and constants not of order n with first-order terms. For example, $x : (\alpha \to \alpha) \to \alpha$ is replaced by $\lambda x_1.z$ where $z : \alpha$ and $\mathsf{g} : \alpha \to \alpha \to \alpha$ is replaced by $\lambda x_1 x_2.z$. Normalise the result. □

Adding Constants

$$\mathbf{U}_n^{\mathcal{C}} \preceq \mathbf{U}_n^{\mathcal{D}} \quad \text{if } \mathcal{C} \subseteq \mathcal{D}$$

Proof Sketch.
Replace constants $d \in \mathcal{D} - \mathcal{C}$ with first-order terms, see conservativity. □

Removing Constans

$$\mathbf{U}_n^{\mathcal{D}} \preceq \mathbf{U}_n^{\mathcal{C}} \quad \text{if } \mathcal{C} \subseteq \mathcal{D} \text{ and } \forall d \notin C.\ \text{ord}\ (\Omega d) < n$$

Proof Sketch.
Let $\mathcal{C} = \{\mathsf{g}\}$ and $\mathcal{D} = \{\mathsf{a}, \mathsf{g}\}$.

$$\begin{array}{lcl} \mathsf{g}\ x \stackrel{?}{=} \mathsf{g}\ \mathsf{a} & & \lambda x_{\mathsf{a}}.\mathsf{g}\ (x\ x_{\mathsf{a}}) \stackrel{?}{=} \lambda x_{\mathsf{a}}.\mathsf{g}\ x_{\mathsf{a}} \\ & \rightsquigarrow & \\ \text{where} \quad x : \alpha & & \text{where} \quad x : \alpha \to \alpha \end{array}$$

□