Problemen/UWC

Universitaire Wiskunde Competitie

De Universitaire Wiskunde Competitie (UWC) is een ladderwedstrijd voor studenten. De uitslagen worden tevens gepubliceerd op de internetpagina http://uwc.ewi.tudelft.nl/uitslag/uitslag.pdf

Ieder nummer bevat de ladderopgaven A, B, en C waarvoor respectievelijk 30, 40 en 50 punten kunnen worden behaald. Daarnaast zijn er respectievelijk 6, 8 en 10 extra punten te winnen voor elegantie en generalisatie. Er worden drie editieprijzen toegekend, van 100, 50, en 25 euro. De puntentotalen van winnaars tellen voor 0, 50, en 75 procent mee in de laddercompetitie. De aanvoerder van de ladder ontvangt een prijs van 100 euro en begint daarna weer onderaan. Daarnaast wordt twee maal per jaar een ster-opgave aangeboden waarvan de redactie geen oplossing bekend is. Voor de eerst ontvangen correcte oplossing van deze ster-opgave wordt eveneens 100 euro toegekend.
Groepsinzendingen zijn toegestaan. Elektronische inzending in LaTeX wordt op prijs gesteld. De inzendtermijn voor de oplossingen sluit op 1 februari 2005. Voor een ster-opgave geldt een inzendtermijn van een jaar.
De Universitaire Wiskunde Competitie wordt gesponsord door Optiver Derivatives Trading en wordt tevens ondersteund door bijdragen van de Nederlandse Onderwijs Commissie voor Wiskunde en de Vereniging voor Studie- en Studentenbelangen te Delft.

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Problem A

1. Show that there exist infinitely many $n \in \mathbf{N}$, such that $S_n = 1 + 2 + \ldots + n$ is a square.
2. Let $a_1, a_2, a_3, \ldots$ be those squares. Calculate $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Problem B

Let G be a finite set of elements and $\cdot$ a binary associative operation on G. There is a neutral element in G and that is the only element in G with the property $a \cdot a = a$. Show that G with the operation $\cdot$ is a group.

Problem C

Let $\{a_n\}_n$ be a sequence ($n \geq 0$), with $a_n \in \{\pm 1\}$ for all n. Define

$$S_n = \sum_{k=0}^{n} a_k a_{n-k}.$$

Prove that $\exists C > 0 : \forall m > 0 : \exists n > m : |S_n| > C\sqrt{n}$.

Edition 2004/1

Op de ronde 2004/1 van de Universitaire Wiskunde Competitie ontvingen we inzendingen van Syb Botma, Kenny De Commer, Filip Cools and Hendrik Hubrechts.

Problem 2004/1-A

For every integer $n > 2$ prove that $\sum_{j=1}^{n-1} \left(1/(n-j) \sum_{k=j}^{n-1} 1/k \right) < \pi^2/6$.

Solution This problem has been solved by Syb Botma, Kenny De Commer, Filip Cools, Klaas Pieter Hart, Hendrik Hubrechts, Ruud Jeurissen and Jaap Spies. Ruud Jeurissen's solution is given here.
Let A_{n-1} denote the left hand side. Changing the order of summation we get

$$A_{n-1} = \sum_{k=1}^{n-1} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{n-j} = \sum_{k=1}^{n-1} \frac{1}{k} \sum_{s=n-k}^{n-1} \frac{1}{s}.$$

Then

$$A_n - A_{n-1} = \sum_{k=1}^{n} \frac{1}{k} \sum_{s=n+1-k}^{n} \frac{1}{s} - \sum_{k=1}^{n-1} \frac{1}{k} \sum_{s=n-k}^{n-1} \frac{1}{s} = \frac{1}{n} \sum_{s=1}^{n} \frac{1}{s} + \sum_{k=1}^{n-1} \frac{1}{k}\Big(\frac{1}{n} - \frac{1}{n-k}\Big)$$
$$= \frac{1}{n} \sum_{s=1}^{n} \frac{1}{s} - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{n-k} = \frac{1}{n}\Big(\sum_{s=1}^{n} \frac{1}{s} - \sum_{t=1}^{n-1} \frac{1}{t}\Big) = \frac{1}{n^2}.$$

Since $A_1 = 1$, we find $A_n = \sum_{i=1}^{n} \frac{1}{i^2} < \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$ (and $\lim_{n\to\infty} A_{n-1} = \frac{\pi^2}{6}$).

Problem 2004/1-B

Consider the first digits of the numbers 2^n: $1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, \ldots$. Does the digit 7 appear in this sequence? Which digit appears more often, 7 or 8? How many times more often?

Solution This problem has been solved by Filip Cools, Hendrik Hubrechts, Kenny De Commer, Syb Botma and Jaap Spies. The problem has been taken from V.I. Arnold's *Mathematical methods in classical mechanics* (it is the final problem of section 16 and its solution is given in section 51). By coincidence a solution of the problem appeared in the June 2004 issue of the Newsletter of the European Mathematical Society. The answers are: 7 does occur (though it takes some time before its first occurence) and it occurs more often than 8. To specify the answer to the third question one has to use an averaging result (e.g. the ergodic theorem or Weyl's criterion).
Take logarithms to find that the first digit of 2^n is equal to 7 if and only if

$$\log 7 \le n \log 2 < \log 8, \ \bmod 1$$

Represent the circle by the real numbers modulo 1. The map $x \mapsto x + \log 2$ is an irrational rotation ρ of the circle. The equation above now says that 7 occurs as the first digit of 2^n if and only if the unit element of the circle rotates into $(\log 7, \log 8)$ under ρ^n. It is known that an irrational rotation is uniquely ergodic and by the ergodic theorem, for numbers $a < b$ in $(0, 1)$ the fraction

$$\frac{|\{n \ \le N\colon \rho^n(x) \in (a, b) \bmod 1\}|}{N}$$

converges to $b - a$ as N goes to infinity, regardless of the choice of x. It follows that the fraction of iterates 2^n with initial digit 7 is equal to $\log 8 - \log 7$ as N goes to infinity. Similarly, the fraction of iterates with initial digit 8 is equal to $\log 9 - \log 8$.

Problem 2004/1-C

We have a circular key chain and we want to colour the keys, using as few colours as possible, so that each key can be identified by the color pattern — that is, by looking at the key's colour and neighboring colours as far away as needed. Let $f(n)$ be the minimal number of colours required to uniquely disambiguate a circular key chain of n keys in this way. Determine $f(n)$ for all positive integers n.

Solution This problem has been solved by Filip Cools, Hendrik Hubrechts, Kenny De Commer and Ruud Jeurissen. It is problem 729 in the Journal of Recreational Mathematics (vol 11, 1979), proposed by Frank Rubin. It has appeared in many puzzle corners ever after and it has sprouted the 'distinguishing number' in graph theory. The answer is that $f(n) = 2$ if $n \ge 6$ and $f(n) = 3$ if $n = 3, 4, 5$.
Enumerate the keys $1, 2, 3, \ldots, n$ cyclically. To disambiguate the chain you have to be able to find keys 1 and 2, since then you can find the other keys by counting. So $f(n) \le 3$, since you can colour 1 green, 2 red and colour the others yellow. Suppose $n \ge 6$. Colour $1, 3, n$ red and colour all other keys green. Then 2 is the only green key that has two red neigbours since $n \ge 6$. So you can find key 2 and key 1 is its only neighbour that has a red neighbour.

Oplossingen

We need to show that $f(n) \neq 2$ if $n = 3, 4, 5$. Suppose we can use red and green to disambiguate the key chain. Clearly it does not suffice to colour one key differently from all the others. Since $n = 3, 4, 5$ we may assume that two keys are green while the others are red. Turn the key ring around (reflect) in such a way that the green keys exchange their position. This action does not change the colour pattern, so the key ring cannot be disambiguated.
Kenny De Commer proposes and solves a related key chain problem: what is the minimal number of colours that you need in order to disambiguate the chain, regardless of how you apply the colours?

Problem 2003/1-B
Jos Brands has pointed out that the solution to UWC problem 2003/1-B as given in last year's September issue is not correct. We will come back to this problem in a later issue.

In the previous issue we remarked that Bert Jagers gave a general solution to problem 2003/4-B, which is presented here.

Problem 2003/4-B
Let and m and n be coprime. Assume that G is a group such that m-th powers and n-th powers commute. Then G is abelian.

Solution Let $M \subset G$ be generated by all m-th powers and let $N \subset G$ be generated by all n-th powers. These subgroups are clearly invariant under automorphisms, hence they are normal. Since m and n are coprime $G = MN$ and $M \cap N$ is contained in the center of G. Let $s \in M$ and $t \in N$ be arbitrary elements. To settle that G is abelian it suffices to show that $st = ts$, in other words, the commutator $[s, t]$ is equal to e . Observe that $[s, t] = sts^{-1}t^{-1} \in M \cap N$ since, by normality, $[s, t]$ is a product of two elements of M as well as a product of two elements of N. Hence $[s, t]$ is an element of the center, say $[s, t] = z$. In other words $sts^{-1} = zt$, so $st^ms^{-1} = z^mt^m$. Since $t^m \in N$ it commutes with s, so $z^m = e$. In exactly the same way $z^n = e$. So $z = [s, t] = e$.

Uitslag Editie 2004/1

	Naam	A	B	C	*Totaal*
1.	Hendrik Hubrechts	11	9	10	119
2.	Filip Cools	10	8	10	112
3.	Kenny De Commer	10	3	11	97

Ladderstand Universitaire Wiskunde Competitie
We vermelden alleen de top 5. Voor de complete ladderstand verwijzen we naar de UWC-website.

	Naam	*Punten*
1.	Kenny De Commer	142
2.	Tom Claeys	138
3.	Gerben Stavenga e.a.	136
4.	Filip Cools e.a.	107
5.	Peter Bruin	99