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STOCHASTIC FUZZY DIFFERENTIAL EQUATIONS WITH AN APPLICATION

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In this paper we present the existence and uniqueness of solutions to the stochastic fuzzy differential equations driven by Brownian motion. The continuous dependence on initial condition and stability properties are also established. As an example of application we use some stochastic fuzzy differential equation in a model of population dynamics.

Keywords: fuzzy random variable, fuzzy stochastic process, fuzzy stochastic Lebesgue–Aumann integral, fuzzy stochastic Itô integral, stochastic fuzzy differential equation, stochastic fuzzy integral equation

Classification: 60H05, 60H10, 60H30, 03E72

1. INTRODUCTION

The theory of fuzzy differential equations has focused much attention in the last decades since it provides good models for dynamical systems under uncertainty. Kaleva (in his paper [8]) started to develop this theory using the concept of H-differentiability for fuzzy mappings introduced by Puri and Ralescu [18]. Currently the literature on this topic is very rich. For a significant collection of the results on fuzzy differential equations and further references we refer the reader to the monographs of Lakshmikantham and Mohapatra [11], Diamond and Kloeden [3].

Recently some results have been published concerning random fuzzy differential equations (see Fei [4], Feng [5], Malinowski [13]). The random approach can be adequate in modeling of the dynamics of real phenomena which are subjected to two kinds of uncertainty: randomness and fuzziness, simultaneously. Here a crucial role play fuzzy random variables and fuzzy stochastic processes. In literature one can find various definitions of fuzzy random variables as well as the results which establish the relations between different concepts of measurability for fuzzy random elements (see e. g. Colubi et al. [2]).

In [13] there were investigated the random fuzzy differential equations which, in their integral form, contain random fuzzy Lebesgue–Aumann integral. The results such as existence, uniqueness of the solutions to these equations were shown. Also some applications of random fuzzy differential equations in the real-world phenomena were presented. The extension of these studies and the next step in modeling of dynamical systems under two types of uncertainties should be the theory of stochastic fuzzy differential equations in which the stochastic fuzzy diffusion term (stochastic fuzzy Itô integral) appears. The crisp stochastic differential equations with stochastic perturbation terms are successfully used in a great number of mathematical description of real phenomena in control theory, physics, economics, biology (see e. g. Øksendal [16], Protter [17] and references therein). The models involving stochastic fuzzy differential equations could be promising in the framework of phenomena where the quantities have imprecise values.

As far as we know there are two papers concerning this new area, i. e. Kim [9] and Ogura [15]. However the approaches presented there are different. In [9] all the considerations are made in the setup of fuzzy sets space of a real line, and the main result on the existence and uniquenes of the solution is obtained under very particular conditions imposed on the structure of integrated fuzzy stochastic processes such that a maximal inequality for fuzzy stochastic Itô integrals holds. Unfortunately the paper [9] contains gaps. Moreover, in view of Zhang [21] we find out that the intersection property (a crucial one to apply the *Representation Theorem* of Negoita–Ralescu [14]) of a set-valued Itô integral may not hold true in general. Thus a definition of fuzzy stochastic Itô integral, which is used in [9], seems to be incorrect. Hence, unfortunately, most of results in [9] seem to be questionable. On the other hand, in [15] a proposed approach does not contain any notion of fuzzy stochastic Itô integral. The method presented there is based on selections sets. Therefore, in this paper, we propose a new approach to the notion of fuzzy stochastic Itô integral and consequently a new approach to stochastic fuzzy differential equations. We give a result of existence and uniqueness of the solution to stochastic fuzzy differential equation where the diffusion term (appropriate fuzzy stochastic Itô integral) is of some special form, i. e. it is the embedding of real d-dimensional Itô integral into fuzzy numbers space. We impose only standard requirements on the equation coefficients, i. e. the Lipschitz condition and a linear growth condition. The existence theorem is obtained in the framework of a space of L^2-integrably bounded fuzzy random variables which is complete with respect to the considered metric. Further we examine a boundedness of the solution, a continuous dependence on the initial conditions and a stability of solutions.

The paper is organized as follows: in Section 2 we give some preliminaries on measurable multifunctions and fuzzy random variables, which we will need later on. In Section 3 the notions of fuzzy stochastic integrals of Lebesgue–Aumann type and Itô type are defined, also some useful properties of these integrals are stated. In Section 4 the stochastic fuzzy differential equations are investigated, and in Section 5 we apply them to a model of population dynamics.

2. PRELIMINARIES

Let $\mathcal{K}(\mathbb{R}^d)$ be the family of all nonempty, compact and convex subsets of $\mathbb{R}^d$. In $\mathcal{K}(\mathbb{R}^d)$ we consider the Hausdorff metric d_H which is defined by

$$d_H\,(A,B) := \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\},$$

where $\|\cdot\|$ denotes a norm in $I\!R^d$. It is known that $\mathcal{K}(I\!R^d)$ is a complete and separable metric space with respect to d_H.

If $A, B, C \in \mathcal{K}(I\!R^d)$, we have $d_H(A + C, B + C) = d_H(A, B)$ (see e. g. Lakshmikantham, Mohapatra [11]).

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $\mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(I\!R^d))$ denote the family of $\mathcal{A}$-measurable multifunctions with values in $\mathcal{K}(I\!R^d)$, i. e. the mappings $F : \Omega \to \mathcal{K}(I\!R^d)$ such that

$$\{\, \omega \in \Omega : F(\omega) \cap C \neq \emptyset \,\} \in \mathcal{A} \quad \text{for every closed set } \ C \subset I\!R^d.$$

A multifunction $F \in \mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(I\!R^d))$ is said to be L^p-integrably bounded, $p \geq 1$, if there exists $h \in L^p(\Omega, \mathcal{A}, P; I\!R_+)$ such that $|||F||| \leq h$ P-a.e., where $I\!R_+ := [0, \infty)$,

$$|||A||| := d_H(A, \{0\}) = \sup_{a \in A} \|a\| \quad \text{for} \quad A \in \mathcal{K}(I\!R^d)$$

and $L^p(\Omega, \mathcal{A}, P; I\!R_+)$ is a space of equivalence classes (with respect to the equality P-a.e.) of $\mathcal{A}$-measurable random variables $h : \Omega \to I\!R_+$ such that $I\!Eh^p = \int_\Omega h^p \, dP < \infty$. It is known (see Hiai and Umegaki [6]) that $F \in \mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(I\!R^d))$ is L^p-integrably bounded if and only if $|||F||| \in L^p(\Omega, \mathcal{A}, P; I\!R_+)$. Let us denote

$$\mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(I\!R^d)) := \Big\{ F \in \mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(I\!R^d)) : |||F||| \in L^p(\Omega, \mathcal{A}, P; I\!R_+) \Big\}.$$

The multifunctions $F, G \in \mathcal{L}^p\big(\Omega, \mathcal{A}, P; \mathcal{K}(I\!R^d)\big)$ are considered to be identical, if $F = G$ P-a.e.

For $F, G \in \mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(I\!R^d))$ there exist sequences $\{f_n\}, \{g_n\}$ of measurable selections for F and G, respectively, such that $F(\omega) = \mathrm{cl}\{f_n(\omega) : n \geq 1\}$ and $G(\omega) = \mathrm{cl}\{g_n(\omega) : n \geq 1\}$, where cl denotes the closure in $I\!R^d$. Hence the function $\omega \mapsto d_H(F(\omega), G(\omega))$ is measurable. Since $d_H(F, G) \leq |||F||| + |||G|||$, we have $d_H(F, G) \in L^p(\Omega, \mathcal{A}, P; I\!R_+)$ for $F, G \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(I\!R^d))$. Therefore one can define the distance

$$\Delta_p(F, G) := \left(I\!E d_H^p(F, G)\right)^{1/p} \quad \text{for} \ \ F, G \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(I\!R^d)), \ p \geq 1.$$

In fact Δ_p is a metric in the set $\mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(I\!R^d))$.

One can prove that:

Theorem 2.1. For $p \geq 1$ the space $\mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(I\!R^d))$ is a complete metric space with respect to the metric Δ_p.

Let $\mathcal{F}(I\!R^d)$ denote the fuzzy set space of $I\!R^d$, i. e. the set of functions $u : I\!R^d \to [0, 1]$ such that $[u]^\alpha \in \mathcal{K}(I\!R^d)$ for every $\alpha \in [0, 1]$, where $[u]^\alpha := \{\, a \in I\!R^d : u(a) \geq \alpha \,\}$ for $\alpha \in (0, 1]$ and $[u]^0 := \mathrm{cl}\{\, a \in I\!R^d : u(a) > 0 \,\}$.

For $u \in \mathcal{F}(I\!R^d)$ we define $\sigma(p^*, \alpha; u) := \sup\{(p^*, a) : a \in [u]^\alpha\}$ and call it the support function of the fuzzy set u at $p^* \in I\!R^d$ and $\alpha \in [0, 1]$, where $(\cdot, \cdot)$ inside of the supremum denotes the inner product in $I\!R^d$.

Definition 2.2. (Puri and Ralescu [19]). Let $(\Omega, \mathcal{A}, P)$ be a probability space. A mapping $x\colon \Omega \to \mathcal{F}(I\!R^d)$ is said to be a fuzzy random variable, if $[x]^\alpha\colon \Omega \to \mathcal{K}(I\!R^d)$ is an $\mathcal{A}$-measurable multifunction for all $\alpha \in [0,1]$.

The following result is a consequence of Proposition 2.39 in chap. 2 of Hu and Papageorgiou [7].

Proposition 2.3. Let $(\Omega, \mathcal{A}, P)$ be a complete probability space. A mapping $x\colon \Omega \to \mathcal{F}(I\!R^d)$ is a fuzzy random variable if and only if for every $\alpha \in [0,1]$ and every $p^* \in I\!R^d$ the function $\Omega \ni \omega \mapsto \sigma\left(p^*, \alpha; x(\omega)\right) \in I\!R$ is $\mathcal{A}$-measurable.

Definition 2.4. A fuzzy random variable $x\colon \Omega \to \mathcal{F}(I\!R^d)$ is said to be L^p-integrably bounded, $p \geq 1$, if $[x]^\alpha \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(I\!R^d))$ for every $\alpha \in [0,1]$.

Let $\mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{F}(I\!R^d))$ denote the set of all the L^p-integrably bounded fuzzy random variables, where we consider $x, y \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{F}(I\!R^d))$ as identical if $P([x]^\alpha = [y]^\alpha,\ \forall \alpha \in [0,1]) = 1$.

Remark 2.5. Let $x\colon \Omega \to \mathcal{F}(I\!R^d)$ be a fuzzy random variable and $p \geq 1$. The following conditions are equivalent:

(a) $x \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{F}(I\!R^d))$,

(b) $[x]^0 \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(I\!R^d))$,

(c) $\|[x]^0\| \in L^p(\Omega, \mathcal{A}, P; I\!R_+)$.

By virtue of Proposition 5.2 in chap. 2 of Hu and Papageorgiou [7] we can write the following assertion.

Proposition 2.6. If $x \in \mathcal{L}^1(\Omega, \mathcal{A}, P; \mathcal{F}(I\!R^d))$, then for every $\alpha \in [0,1]$ and every $p^* \in I\!R^d$ it holds

$$\sigma\Big(p^*, \alpha; \int_\Omega x \, \mathrm{d}P\Big) = \int_\Omega \sigma(p^*, \alpha; x) \, \mathrm{d}P,$$

where $\int_\Omega x \, \mathrm{d}P$ is a fuzzy integral defined levelwise in the same manner as in Kaleva [8], i. e. the level sets of this integral are the set-valued integrals of level sets of x in the sense of Aumann [1].

For $x, y \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{F}(I\!R^d))$ the mapping $\omega \mapsto d_H^p([x(\omega)]^\alpha, [y(\omega)]^\alpha)$ is $\mathcal{A}$-measurable for every $\alpha \in [0,1]$. Moreover, we have

$$\begin{aligned}
\sup_{\alpha \in [0,1]} \Delta_p([x]^\alpha, [y]^\alpha) &\leq \sup_{\alpha \in [0,1]} \Delta_p([x]^\alpha, \{0\}) + \sup_{\alpha \in [0,1]} \Delta_p([y]^\alpha, \{0\}) \\
&\leq \Big(I\!E \sup_{\alpha \in [0,1]} d_H^p([x]^\alpha, \{0\})\Big)^{1/p} + \Big(I\!E \sup_{\alpha \in [0,1]} d_H^p([y]^\alpha, \{0\})\Big)^{1/p} \\
&\leq \Delta_p([x]^0, \{0\}) + \Delta_p([y]^0, \{0\}) < \infty.
\end{aligned}$$

Therefore we can define a metric in $\mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{F}(I\!R^d))$ in the following way

$$\delta_p(x, y) := \sup_{\alpha \in [0,1]} \Delta_p([x]^\alpha, [y]^\alpha).$$

Remark 2.7. Let $x, y \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{F}(\mathbb{R}^d))$, $p \geq 1$. Then $\delta_p(x, y) = 0$ if and only if $P([x]^\alpha = [y]^\alpha,\ \forall \alpha \in [0, 1]) = 1$.

In a similar way as in the proof of Theorem 1 in Stojaković [20] we proceed with a derivation of the following result.

Theorem 2.8. For $p \geq 1$ the space $\mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{F}(\mathbb{R}^d))$ is a complete metric space with respect to the metric δ_p.

In the subsequent section we will apply the following properties of the metric δ_2 which are immediate after-effects of the properties of the Hausdorff metric (see [7]).

Lemma 2.9. (a) If $x, y, z \in \mathcal{L}^2(\Omega, \mathcal{A}, P; \mathcal{F}(\mathbb{R}^d))$, then

$$\delta_2(x + z, y + z) = \delta_2(x, y). \tag{1}$$

(b) If $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in \mathcal{L}^2(\Omega, \mathcal{A}, P; \mathcal{F}(\mathbb{R}^d))$, then

$$\delta_2^2\Big(\sum_{k=1}^n x_k, \sum_{k=1}^n y_k\Big) \leq n \sum_{k=1}^n \delta_2^2(x_k, y_k). \tag{2}$$

3. FUZZY STOCHASTIC PROCESSES AND FUZZY STOCHASTIC INTEGRALS

In this section we establish the notion of a fuzzy stochastic Lebesgue–Aumann integral as a fuzzy adapted stochastic process with values in the fuzzy set space of d-dimensional Euclidean space. We make also a discussion on a fuzzy stochastic Itô integral.

Let $T \in (0, \infty)$ and let $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t \in [0,T]}, P)$ be a complete, filtered probability space with a filtration $\{\mathcal{A}_t\}_{t \in [0,T]}$ satisfying the usual hypotheses, i. e. $\{\mathcal{A}_t\}_{t \in [0,T]}$ is an increasing and right continuous family of sub-σ-algebras of $\mathcal{A}$, and $\mathcal{A}_0$ contains all P-null sets.

We call $x\colon [0, T] \times \Omega \to \mathcal{F}(\mathbb{R}^d)$ a fuzzy stochastic process, if for every $t \in [0, T]$ a mapping $x(t, \cdot) = x(t)\colon \Omega \to \mathcal{F}(\mathbb{R}^d)$ is a fuzzy random variable in the sense of Definition 2.2, i. e. x can be thought as a family $\{x(t), t \in [0, T]\}$ of fuzzy random variables. A fuzzy stochastic process x is said to be $\{\mathcal{A}_t\}$-adapted, if for every $\alpha \in [0, 1]$ the multifunction $[x(t)]^\alpha\colon \Omega \to \mathcal{K}(\mathbb{R}^d)$ is $\mathcal{A}_t$-measurable for all $t \in [0, T]$. It is called measurable, if $[x]^\alpha\colon [0, T] \times \Omega \to \mathcal{K}(\mathbb{R}^d)$ is a $\mathcal{B}([0, T]) \otimes \mathcal{A}$-measurable multifunction for all $\alpha \in [0, 1]$, where $\mathcal{B}([0, T])$ denotes the Borel σ-algebra of subsets of $[0, T]$. If $x\colon [0, T] \times \Omega \to \mathcal{F}(\mathbb{R}^d)$ is $\{\mathcal{A}_t\}$-adapted and measurable, then it will be called nonanticipating. Equivalently, x is nonanticipating if and only if for every $\alpha \in [0, 1]$ the multifunction $[x]^\alpha$ is measurable with respect to the σ-algebra $\mathcal{N}$, which is defined as follows

$$\mathcal{N} := \{A \in \mathcal{B}([0, T]) \otimes \mathcal{A} : A^t \in \mathcal{A}_t \text{ for every } t \in [0, T]\},$$

where $A^t = \{\omega : (t, \omega) \in A\}$ for $t \in [0, T]$.

Let $p \geq 1$ and $L^p([0,T] \times \Omega, \mathcal{N}; \mathbb{R}^d)$ denote the set of all nonanticipating $\mathbb{R}^d$-valued stochastic processes $\{h(t), t \in [0,T]\}$ such that $\mathbb{E}\Big(\int_0^T \|h(s)\|^p \,\mathrm{d}s\Big) < \infty$. A fuzzy stochastic process x is called L^p-integrably bounded, if there exists a real-valued stochastic process $h \in L^p([0,T] \times \Omega, \mathcal{N}; \mathbb{R}_+)$ such that $\|\|[x(t,\omega)]^0\|\| \leq h(t,\omega)$ for a.a. $(t,\omega) \in [0,T] \times \Omega$. By $\mathcal{L}^p([0,T] \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$ we denote the set of nonanticipating and L^p-integrably bounded fuzzy stochastic processes.

Let $x \in \mathcal{L}^1([0,T] \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$. For such x and a fixed $t \in [0,T]$ we can define an integral

$$L_x(t,\omega) := \int_0^t x(s,\omega)\,\mathrm{d}s$$

depending on the parameter $\omega \in \Omega$, where the fuzzy integral $\int_0^t x(s,\omega)\,\mathrm{d}s$ is defined levelwise, i. e. the α-level sets of this integral are the set-valued integrals of α-level sets of x in the sense of Aumann [1]. For the details and properties of such a fuzzy integral we refer to Kaleva [8]. Since for every $\alpha \in [0,1]$, every $t \in [0,T]$ and every $\omega \in \Omega$ the Aumann integral $\int_0^t [x(s,\omega)]^\alpha \,\mathrm{d}s$ belongs to $\mathcal{K}(\mathbb{R}^d)$ (see e. g. Aumann [1], Kisielewicz [10]), we have $\int_0^t x(s,\omega)\,\mathrm{d}s \in \mathcal{F}(\mathbb{R}^d)$ for every $t \in [0,T]$ and every $\omega \in \Omega$. We will call $L_x(t) = L_x(t,\cdot)$ the fuzzy stochastic Lebesgue–Aumann integral. Obviously, such integral can be defined for every fuzzy stochastic process $x \in \mathcal{L}^p([0,T] \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$, $p \geq 1$.

Proposition 3.1. Let $p \geq 1$ and $x \in \mathcal{L}^p([0,T] \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$. Then the mapping $L_x(\cdot,\cdot)\colon [0,T] \times \Omega \to \mathcal{F}(\mathbb{R}^d)$ is a measurable fuzzy stochastic process and $L_x(t) = L_x(t,\cdot) \in \mathcal{L}^p(\Omega, \mathcal{A}_t, P; \mathcal{F}(\mathbb{R}^d))$ for every $t \in [0,T]$.

Proof. Let us fix $\alpha \in [0,1]$ and $p^* \in \mathbb{R}^d$. Accordingly to the Proposition 2.3 the function $[0,T] \times \Omega \ni (t,\omega) \mapsto \sigma(p^*, \alpha; x(t,\omega)) \in \mathbb{R}$ is measurable and $\{\mathcal{A}_t\}$-adapted. Note that for every $(t,\omega) \in [0,T] \times \Omega$

$$\begin{aligned} \sigma(p^*, \alpha; x(t,\omega)) &= \sup\{\, (p^*, a) : a \in [x(t,\omega)]^\alpha \,\} \\ &\leq \sup\{\, \|p^*\| \cdot \|a\| : a \in [x(t,\omega)]^\alpha \,\} = \|p^*\| \cdot \|\|[x(t,\omega)]^\alpha\|\|. \end{aligned}$$

Hence $\sigma(p^*, \alpha; x(\cdot,\cdot))$ belongs to $L^p([0,T] \times \Omega, \mathcal{N}; \mathbb{R})$.

Using Fubini's theorem we get that the mapping $\omega \mapsto \int_0^t \sigma(p^*, \alpha; x(s,\omega))\,\mathrm{d}s$ is $\mathcal{A}_t$-measurable for every $t \in [0,T]$, and $t \mapsto \int_0^t \sigma(p^*, \alpha; x(s,\omega))\,\mathrm{d}s$ is continuous for $\omega \in \Omega$. By Proposition 2.6 we have $\sigma(p^*, \alpha; \int_0^t x(s,\omega)\,\mathrm{d}s) = \int_0^t \sigma(p^*, \alpha; x(s,\omega))\,\mathrm{d}s$, what allows us to claim that $(t,\omega) \mapsto \sigma(p^*, \alpha; \int_0^t x(s,\omega)\,\mathrm{d}s)$ is a measurable and $\{\mathcal{A}_t\}$-adapted real valued stochastic process. Now by virtue of Proposition 2.3 we infer that the process $[0,T] \times \Omega \ni (t,\omega) \mapsto \int_0^t x(s,\omega)\,\mathrm{d}s \in \mathcal{F}(\mathbb{R}^d)$ is nonanticipating, i. e. it is measurable and $\{\mathcal{A}_t\}$-adapted.

Since $x \in \mathcal{L}^p([0,T] \times \Omega, \mathcal{N}; \mathcal{F}(\mathbb{R}^d))$, there exists $h \in L^p([0,T] \times \Omega, \mathcal{N}; \mathbb{R}_+)$ such that $\|\|[x(t,\omega)]^0\|\| \leq h(t,\omega)$ for a.a. $(t,\omega) \in [0,T] \times \Omega$. Let $t \in [0,T]$ be fixed. Applying Jensen's inequality we obtain

$$\mathbb{E}\Big(\int_0^t h(s)\,\mathrm{d}s\Big)^p \leq t^{p-1}\mathbb{E}\Big(\int_0^t h^p(s)\,\mathrm{d}s\Big) < \infty.$$

Hence $\int_0^t h(s)\,\mathrm{d}s \in L^p(\Omega, \mathcal{A}_t, P; I\!R_+)$. Further, observe that using Th. 4.1. of Hiai and Umegaki [6] we can write

$$\begin{aligned} |||[L_x(t)]^0||| &= d_H\Big(\int_0^t [x(s)]^0 \,\mathrm{d}\,\mathrm{d}s, \{0\}\Big) \\ &\le \int_0^t d_H\big([x(s)]^0, \{0\}\big)\,\mathrm{d}s = \int_0^t |||[x(s)]^0|||\,\mathrm{d}s \le \int_0^t h(s)\,\mathrm{d}s. \end{aligned}$$

By Remark 2.5 the proof is completed. □

Similar reasoning yields the following properties.

Proposition 3.2. Let $x, y \in \mathcal{L}^1([0,T]\times\Omega, \mathcal{N}; \mathcal{F}(I\!R^d))$. Then for every $p \ge 1$ and every $t \in [0,T]$

$$\delta_p^p\big(L_x(t), L_y(t)\big) \le t^{p-1}\int_0^t \delta_p^p\big(x(s), y(s)\big)\,\mathrm{d}s. \tag{3}$$

Moreover, if $x, y \in \mathcal{L}^p([0,T]\times\Omega, \mathcal{N}; \mathcal{F}(I\!R^d))$ with $p \ge 1$ then the right-hand side of the inequality (3) is bounded and the mapping

$$[0,T] \ni t \mapsto L_x(t) \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{F}(I\!R^d))$$

is δ_p-continuous.

In the sequel we shall introduce a concept of a fuzzy stochastic Itô integral (being a fuzzy random variable) needed in the paper.

Firstly, observe that a natural way to define fuzzy Itô integral could be the following one: to define a stochastic set-valued Itô integral (being a measurable multifunction) and then using the *Representation Theorem* of Negoita–Ralescu [14] to introduce a notion of fuzzy Itô integral. Such a method of defining of fuzzy Itô integral one can find in [9, 12]. Unfortunately, this approach fails as we find out from [21] that an intersection property (a crucial one to apply *Representation Theorem*) of the set-valued Itô integral may not hold true in general. As a consequence, this way of defining of fuzzy stochastic Itô integral seems to be incorrect. Therefore the notion of a fuzzy stochastic Itô integral, proposed in this paper, will be of a very particular form.

Let $\langle\cdot\rangle\colon I\!R^d \to \mathcal{F}(I\!R^d)$ denote an embedding of $I\!R^d$ into $\mathcal{F}(I\!R^d)$, i. e. for $r \in I\!R^d$ we have

$$\langle r\rangle(a) = \begin{cases} 1, & \text{if} \quad a = r, \\ 0, & \text{if} \quad a \in I\!R^d \setminus \{r\}. \end{cases}$$

If $x\colon \Omega \to I\!R^d$ is an $I\!R^d$-valued random variable on a probability space $(\Omega, \mathcal{A}, P)$, then $\langle x\rangle\colon \Omega \to \mathcal{F}(I\!R^d)$ is a fuzzy random variable. For stochastic processes we have a similar property.

Remark 3.3. Let $x\colon [0,T]\times\Omega \to I\!R^d$ be an $I\!R^d$-valued stochastic process ($\{\mathcal{A}_t\}$-adapted, measurable, respectively). Then $\langle x\rangle\colon [0,T]\times\Omega \to \mathcal{F}(I\!R^d)$ is a fuzzy stochastic process ($\{\mathcal{A}_t\}$-adapted, measurable, respectively).

In forthcoming section we want to consider the stochastic fuzzy differential equations with a diffusion term which is based on a notion of fuzzy stochastic Itô integral. Let us introduce this fuzzy stochastic integral.

Let $\{B(t), t \in [0,T]\}$ be a one-dimensional $\{\mathcal{A}_t\}$-Brownian motion defined on a complete probability space $(\Omega, \mathcal{A}, P)$ with a filtration $\{\mathcal{A}_t\}_{t\in[0,T]}$ satisfying usual hypotheses. For $x \in L^2([0,T] \times \Omega, \mathcal{N}; I\!R^d)$ let $\int_0^T x(s)\,\mathrm{d}B(s)$ denote the classical stochastic Itô integral (see e. g. [16, 17]).

Definition 3.4. By fuzzy stochastic Itô integral we mean the fuzzy random variable $\big\langle \int_0^T x(s)\,\mathrm{d}B(s) \big\rangle$.

For every $t \in [0,T]$ one can consider the fuzzy stochastic Itô integral $\big\langle \int_0^t x(s)\,\mathrm{d}B(s) \big\rangle$, which is understood in the sense:

$$\Big\langle \int_0^t x(s)\,\mathrm{d}B(s) \Big\rangle := \Big\langle \int_0^T \mathbf{1}_{[0,t]}(s) x(s)\,\mathrm{d}B(s) \Big\rangle,$$

where $\mathbf{1}_{[0,t]}(s) = 1$ if $s \in [0,t]$ and $\mathbf{1}_{[0,t]}(s) = 0$ if $s \in (t,T]$.

Proposition 3.5. Let $x \in L^2([0,T]\times\Omega, \mathcal{N}; I\!R^d)$. Then $\Big\{ \big\langle \int_0^t x(s)\,\mathrm{d}B(s) \big\rangle, t \in [0,T] \Big\}$ is an $\{\mathcal{A}_t\}$-adapted fuzzy stochastic process. Moreover, for every $t \in [0,T]$ we have

$$\Big\langle \int_0^t x(s)\,\mathrm{d}B(s) \Big\rangle \in \mathcal{L}^2(\Omega, \mathcal{A}, P; \mathcal{F}(I\!R^d)).$$

Straightforward calculations and classical Itô isometry yield the next result, which will be useful in the further section.

Proposition 3.6. Let $x, y \in L^2([0,T] \times \Omega, \mathcal{N}; I\!R^d)$. Then for every $t \in [0,T]$

$$\delta_2^2\Big(\Big\langle \int_0^t x(s)\,\mathrm{d}B(s) \Big\rangle, \Big\langle \int_0^t y(s)\,\mathrm{d}B(s) \Big\rangle \Big) = \int_0^t \delta_2^2\big(\langle x(s) \rangle, \langle y(s) \rangle \big)\,\mathrm{d}s, \tag{4}$$

and the mapping

$$[0,T] \ni t \mapsto \Big\langle \int_0^t x(s)\,\mathrm{d}B(s) \Big\rangle \in \mathcal{L}^2(\Omega, \mathcal{A}, P; \mathcal{F}(I\!R^d))$$

is δ_2-continuous.

4. STOCHASTIC FUZZY DIFFERENTIAL EQUATIONS

Let $0 < T < \infty$ and let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a filtration $\{\mathcal{A}_t\}_{t\in[0,T]}$ satisfying usual conditions. By $\{B(t), t \in [0,T]\}$ we denote a one-dimensional $\{\mathcal{A}_t\}$-Brownian motion defined on $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t\in[0,T]}, P)$.

In this paragraph we shall consider the stochastic fuzzy differential equations which can be written in symbolic form as:

$$\mathrm{d}x(t) = f(t, x(t))\,\mathrm{d}t + \big\langle g(t, x(t))\,\mathrm{d}B(t) \big\rangle, \quad x(0) = x_0, \tag{5}$$

where $f\colon [0,T] \times \Omega \times \mathcal{F}(I\!R^d) \to \mathcal{F}(I\!R^d)$, $g\colon [0,T] \times \Omega \times \mathcal{F}(I\!R^d) \to I\!R^d$, and $x_0\colon \Omega \to \mathcal{F}(I\!R^d)$ is a fuzzy random variable.

Definition 4.1. By a solution to (5) we mean a fuzzy stochastic process $x\colon [0,T]\times\Omega\to\mathcal{F}(\mathbb{R}^d)$ such that

(i) $x(t)\in\mathcal{L}^2(\Omega,\mathcal{A}_t,P;\mathcal{F}(\mathbb{R}^d))$ for every $t\in[0,T]$,

(ii) $x\colon [0,T]\to\mathcal{L}^2(\Omega,\mathcal{A},P;\mathcal{F}(\mathbb{R}^d))$ is a continuous mapping with respect to the metric δ_2,

(iii) for every $t\in[0,T]$ it holds

$$x(t)=x_0+\int_0^t f(s,x(s))\,\mathrm{d}s+\Big\langle\int_0^t g(s,x(s))\,\mathrm{d}B(s)\Big\rangle\ \ P\text{-a.e.}\tag{6}$$

The right-hand side of (6) is understood in the meaning described in the preceding section, i. e. the second term is the fuzzy stochastic Lebesgue–Aumann integral, while the third one is the $\mathbb{R}^d$-valued stochastic Itô integral which is embedded into $\mathcal{F}(\mathbb{R}^d)$.

Definition 4.2. A solution $x\colon [0,T]\times\Omega\to\mathcal{F}(\mathbb{R}^d)$ to (5) is unique, if for every $t\in[0,T]$

$$P\left([x(t)]^\alpha=[y(t)]^\alpha,\ \ \forall\alpha\in[0,1]\right)=1,$$

where $y\colon [0,T]\times\Omega\to\mathcal{F}(\mathbb{R}^d)$ is any solution of (5).

Here the concepts of solution to (6) and its uniqueness are in the weaker sense than those proposed in Kim [9]. In our new setting it is enough to impose only the standard conditions on the random coefficients of the equation in order to obtain both the existence and the uniqueness of the solution. In the sequel we shall write down the detailed conditions imposed on the coefficients of the equation (5). However, first, we recall some needed facts about different measurability concepts for fuzzy random elements. As we mentioned in the Introduction, the Definition 2.2 is one of the possible to be considered for fuzzy random variables. Generally, having a metric ρ in the set $\mathcal{F}(\mathbb{R}^d)$ one can consider σ-algebra $\mathcal{B}_\rho$ generated by the topology induced by ρ. Then a fuzzy random variable can be viewed as a measurable (in the classical sense) mapping between two measurable spaces, namely $(\Omega,\mathcal{A})$ and $(\mathcal{F}(\mathbb{R}^d),\mathcal{B}_\rho)$. Using the classical notation, we write this as: x is $\mathcal{A}|\mathcal{B}_\rho$-measurable. The metrics which are the most often used in the set $\mathcal{F}(\mathbb{R}^d)$ are:

$$d_\infty(u,v):=\sup_{\alpha\in[0,1]}d_H\big([u]^\alpha,[v]^\alpha\big),$$

$$d_p(u,v):=\Big(\int_0^1 d_H^p\big([u]^\alpha,[v]^\alpha\big)\,\mathrm{d}\alpha\Big)^{1/p},\quad p\geq 1,$$

and Skorohod metric

$$d_S(u,v):=\inf_{\lambda\in\Lambda}\max\left\{\sup_{t\in[0,1]}|\lambda(t)-t|,\ \sup_{t\in[0,1]}d_H(x_u(t),x_v(\lambda(t)))\right\},$$

where Λ denotes the set of strictly increasing continuous functions $\lambda\colon [0,1] \to [0,1]$ such that $\lambda(0) = 0$, $\lambda(1) = 1$, and $x_u, x_v\colon [0,1] \to \mathcal{K}(I\!R^d)$ are the càdlàg representations for the fuzzy sets $u, v \in \mathcal{F}(I\!R^d)$, see Colubi et al. [2] for details. The space $(\mathcal{F}(I\!R^d), d_\infty)$ is complete and non-separable, $(\mathcal{F}(I\!R^d), d_p)$ is separable and non-complete, and the space $(\mathcal{F}(I\!R^d), d_S)$ is Polish.

The fuzzy random variables defined such as in Definition 2.2 will be called Puri–Ralescu fuzzy random variables. It is known (see [2]) that for a mapping $x\colon \Omega \to \mathcal{F}(I\!R^d)$, where $(\Omega, \mathcal{A}, P)$ is a given probability space, it holds:

(v1) x is the Puri–Ralescu fuzzy random variable if and only if x is $\mathcal{A}|\mathcal{B}_{d_S}$-measurable,

(v2) x is the Puri–Ralescu fuzzy random variable if and only if x is $\mathcal{A}|\mathcal{B}_{d_p}$-measurable for all $p \in [1, \infty)$,

(v3) if x is $\mathcal{A}|\mathcal{B}_{d_\infty}$-measurable, then it is the Puri–Ralescu fuzzy random variable; the opposite implication is not true.

Hence the Skorohod metric measurability condition on $\mathcal{F}(I\!R^d)$ is equivalent to the measurability of the α-level mappings and to the $\mathcal{A}|\mathcal{B}_{d_p}$-measurability for all $p \geq 1$.

Now we are in the position to formulate the assumptions imposed on the equation coefficients. Assume that $f\colon [0,T] \times \Omega \times \mathcal{F}(I\!R^d) \to \mathcal{F}(I\!R^d)$, $f \not\equiv \hat{\theta}$, $g\colon [0,T] \times \Omega \times \mathcal{F}(I\!R^d) \to I\!R^d$ satisfy:

(c1) the mapping $f\colon ([0,T] \times \Omega) \times \mathcal{F}(I\!R^d) \to \mathcal{F}(I\!R^d)$ is $\mathcal{N} \otimes \mathcal{B}_{d_S}|\mathcal{B}_{d_S}$-measurable and $g\colon ([0,T] \times \Omega) \times \mathcal{F}(I\!R^d) \to I\!R^d$ is $\mathcal{N} \otimes \mathcal{B}_{d_S}|\mathcal{B}(I\!R^d)$-measurable,

(c2) there exists a constant $L > 0$ such that

$$\delta_2\big(f(t,u), f(t,v)\big) \leq L\delta_2(u,v),$$

$$\big(I\!E\|g(t,u) - g(t,v)\|^2\big)^{1/2} = \delta_2\big(\langle g(t,u)\rangle, \langle g(t,v)\rangle\big) \leq L\delta_2(u,v)$$

for every $t \in [0,T]$, and every $u, v \in \mathcal{F}(I\!R^d)$,

(c3) there exists a constant $C > 0$ such that for every $t \in [0,T]$, and every $u \in \mathcal{F}(I\!R^d)$

$$\delta_2\big(f(t,u), \hat{\theta}\big) \leq C\big(1 + \delta_2(u, \hat{\theta})\big),$$

$$\big(I\!E\|g(t,u)\|^2\big)^{1/2} = \delta_2\big(\langle g(t,u)\rangle, \hat{\theta}\big) \leq C\big(1 + \delta_2(u, \hat{\theta})\big),$$

where $\hat{\theta} \in \mathcal{F}(I\!R^d)$ is defined as $\hat{\theta} := \langle 0\rangle$.

One can see that for non-random u, v the right-hand sides of the inequalities appearing in (c2), (c3) could be written as $Ld_\infty(u,v)$ and $C(1+d_\infty(u,\hat{\theta}))$, respectively. However, in the sequel we will work with u, v which will be random elements, so we keep (c2), (c3) with δ_2 as above.

Using the properties (v1), (v2) and observing that $\mathcal{B}_{d_1} \subset \mathcal{B}_{d_p}$ for all $p \geq 1$, we can rewrite the condition (c1) in its equivalent form as follows:

(c11) the mapping $f\colon ([0,T]\times\Omega)\times\mathcal{F}(I\!R^d)\to\mathcal{F}(I\!R^d)$ is $\mathcal{N}\otimes\mathcal{B}_{d_p}|\mathcal{B}_{d_q}$-measurable for all $p,q\in[1,\infty)$, and $g\colon ([0,T]\times\Omega)\times\mathcal{F}(I\!R^d)\to I\!R^d$ is $\mathcal{N}\otimes\mathcal{B}_{d_1}|\mathcal{B}(I\!R^d)$-measurable,

Each subsequent condition (c12) or (c13) implies that (c1) holds:

(c12) — for every $u\in\mathcal{F}(I\!R^d)$
the mapping $f(\cdot,\cdot,u)\colon [0,T]\times\Omega\to\mathcal{F}(I\!R^d)$ is the nonanticipating fuzzy stochastic process, and $g(\cdot,\cdot,u)\colon [0,T]\times\Omega\to I\!R^d$ is the nonanticipating $I\!R^d$-valued stochastic process,
— for every $(t,\omega)\in[0,T]\times\Omega$
the fuzzy mapping $f(t,\omega,\cdot)\colon \mathcal{F}(I\!R^d)\to\mathcal{F}(I\!R^d)$ is continuous with respect to the metric d_S, and the mapping $g(t,\omega,\cdot)\colon \mathcal{F}(I\!R^d)\to I\!R^d$ is continuous as a function from a metric space $(\mathcal{F}(I\!R^d),d_S)$ to $(I\!R^d,\|\cdot\|)$,

(c13) — for every $u\in\mathcal{F}(I\!R^d)$ the mapping $f(\cdot,\cdot,u)\colon [0,T]\times\Omega\to\mathcal{F}(I\!R^d)$ is the nonanticipating fuzzy stochastic process and $g(\cdot,\cdot,u)\colon [0,T]\times\Omega\to I\!R^d$ is the nonanticipating $I\!R^d$-valued stochastic process,
— for every $(t,\omega)\in[0,T]\times\Omega$
the fuzzy mapping $f(t,\omega,\cdot)\colon \mathcal{F}(I\!R^d)\to\mathcal{F}(I\!R^d)$ is continuous as a mapping from a metric space $(\mathcal{F}(I\!R^d),d_p)$ to $(\mathcal{F}(I\!R^d),d_q)$, for every $p,q\in[1,\infty)$,
the mapping $g(t,\omega,\cdot)\colon \mathcal{F}(I\!R^d)\to I\!R^d$ is continuous as a function from a metric space $(\mathcal{F}(I\!R^d),d_1)$ to $(I\!R^d,\|\cdot\|)$.

Each of the conditions (c1), (c11), (c12), (c13) guarantees the proper measurability of the integrands in (6). In particular, we have:

Lemma 4.3. Let $f\colon [0,T]\times\Omega\times\mathcal{F}(I\!R^d)\to\mathcal{F}(I\!R^d)$, $g\colon [0,T]\times\Omega\times\mathcal{F}(I\!R^d)\to I\!R^d$ satisfy the condition (c1) and a nonanticipating fuzzy stochastic process $x\colon [0,T]\times\Omega\to\mathcal{F}(I\!R^d)$ be given. Then the mapping $f\circ x\colon [0,T]\times\Omega\to\mathcal{F}(I\!R^d)$, $g\circ x\colon [0,T]\times\Omega\to I\!R^d$ defined by

$$(f\circ x)(t,\omega):=f(t,\omega,x(t,\omega)),\quad (g\circ x)(t,\omega):=g(t,\omega,x(t,\omega))$$

for $(t,\omega)\in[0,T]\times\Omega$, is a nonanticipating fuzzy stochastic process and a nonanticipating $I\!R^d$-valued stochastic process, respectively.

Now we formulate the main result of the paper.

Theorem 4.4. Let $x_0\in\mathcal{L}^2(\Omega,\mathcal{A},P;\mathcal{F}(I\!R^d))$ be an $\mathcal{A}_0$-measurable fuzzy random variable and let $f\colon [0,T]\times\Omega\times\mathcal{F}(I\!R^d)\to\mathcal{F}(I\!R^d)$, $g\colon [0,T]\times\Omega\times\mathcal{F}(I\!R^d)\to I\!R^d$ satisfy (c1)–(c3). Then the equation (5) has a unique solution.

P r o o f. We shall prove the theorem in the setup of metric space $\big(\mathcal{L}^2(\Omega,\mathcal{A},P;\mathcal{F}(I\!R^d)),\delta_2\big)$ which is complete due to Theorem 2.8.

Let us define a sequence $x_n\colon [0,T]\times\Omega\to\mathcal{F}(I\!R^d)$, $n=0,1,\dots$ of successive approximations as follows:

$$x_0(t)=x_0,\quad\text{for every}\ \ t\in[0,T],$$

and for $n = 1, 2, \dots$

$$x_n(t) = x_0 + \int_0^t f(s, x_{n-1}(s))\,\mathrm{d}s + \Big\langle \int_0^t g(s, x_{n-1}(s))\,\mathrm{d}B(s) \Big\rangle \quad \text{for every } t \in [0,T].$$

Note that applying (1), (2), (3), (4) we obtain for every $t \in [0,T]$

$$\begin{aligned} \delta_2^2\big(x_1(t), x_0(t)\big) &= \delta_2^2\Big(\int_0^t f(s,x_0)\,\mathrm{d}s + \Big\langle \int_0^t g(s,x_0)\,\mathrm{d}B(s)\Big\rangle, \hat{\theta}\Big) \\ &\le 2\delta_2^2\Big(\int_0^t f(s,x_0)\,\mathrm{d}s, \hat{\theta}\Big) + 2\delta_2^2\Big(\Big\langle \int_0^t g(s,x_0)\,\mathrm{d}B(s)\Big\rangle, \hat{\theta}\Big) \\ &\le 2t\int_0^t \delta_2^2\big(f(s,x_0), \hat{\theta}\big)\,\mathrm{d}s + 2\int_0^t \delta_2^2\big(\langle g(s,x_0)\rangle, \hat{\theta}\big)\,\mathrm{d}s. \end{aligned}$$

Using the assumption (c3) we get

$$\delta_2^2\big(x_1(t), x_0(t)\big) \le 2^2C^2\gamma(T+1)t \le 2^2C^2\gamma(T+1)T < \infty,$$

where $\gamma = 1 + \delta_2^2(x_0, \hat{\theta})$.

Observe further that for $n = 2, 3, \dots$ one has

$$\begin{aligned} \delta_2^2\big(x_n(t), x_{n-1}(t)\big) &\le 2t\int_0^t \delta_2^2\big(f(s,x_{n-1}(s)), f(s,x_{n-2}(s))\big)\,\mathrm{d}s \\ &\quad + 2\int_0^t \delta_2^2\big(\langle g(s,x_{n-1}(s))\rangle, \langle g(s,x_{n-2}(s))\rangle\big)\,\mathrm{d}s. \end{aligned}$$

Hence, using assumption (c2), we infer that

$$\delta_2^2\big(x_n(t), x_{n-1}(t)\big) \le 2L^2(T+1)\int_0^t \delta_2^2\big(x_{n-1}(s), x_{n-2}(s)\big)\,\mathrm{d}s,$$

and therefore

$$\delta_2^2\big(x_n(t), x_{n-1}(t)\big) \le 2L^{-2}C^2\gamma\frac{\big(2L^2(T+1)t\big)^n}{n!} \le 2L^{-2}C^2\gamma\frac{\big(2L^2(T+1)T\big)^n}{n!} < \infty.$$

It follows that $x_n(t) \in \mathcal{L}^2(\Omega, \mathcal{A}_t, P; \mathcal{F}(\mathbb{R}^d))$ for every n and every t. Moreover, for every n the mapping $x_n(\cdot)\colon [0,T] \to \mathcal{L}^2(\Omega, \mathcal{A}, P; \mathcal{F}(\mathbb{R}^d))$ is continuous with respect to the metric δ_2.

In the sequel we shall show that the sequence $(x_n(t))_{n=0}^{\infty}$ satisfies Cauchy condition uniformly in t. Notice that

$$\delta_2\big(x_n(t), x_m(t)\big) \le \big(2L^{-2}C^2\gamma\big)^{1/2}\sum_{k=m+1}^{n}\left(\frac{(2L^2(T+1)T)^k}{k!}\right)^{1/2},$$

and the series $\sum_{k=0}^{\infty}\left(\frac{z^k}{k!}\right)^{1/2}$ is convergent for every $z \in \mathbb{R}$. Hence for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n, m \ge n_0$ it holds

$$\sup_{t\in[0,T]} \delta_2\big(x_n(t), x_m(t)\big) < \varepsilon.$$

Thus $(x_n)_{n=0}^{\infty}$ is uniformly convergent to some fuzzy stochastic process $x\colon [0,T]\times\Omega\to\mathcal{F}(\mathbb{R}^d)$ which is $\{\mathcal{A}_t\}$-adapted and δ_2-continuous. We want to show that this limit process is a solution to (5). In order to do this we show that x satisfies (6). Indeed, for every $t\in[0,T]$ we have

$$\begin{aligned}
&\delta_2^2\Big(x(t), x_0+\int_0^t f(s,x(s))\,\mathrm{d}s+\Big\langle\int_0^t g(s,x(s))\,\mathrm{d}B(s)\Big\rangle\Big)\\
\le\ & 3\delta_2^2\big(x(t),x_n(t)\big)\\
&+3\delta_2^2\Big(x_n(t), x_0+\int_0^t f(s,x_{n-1}(s))\,\mathrm{d}s+\Big\langle\int_0^t g(s,x_{n-1}(s))\,\mathrm{d}B(s)\Big\rangle\Big)\\
&+3\delta_2^2(S_{n-1},S),
\end{aligned}$$

where

$$\begin{aligned}
S_{n-1}&=\int_0^t f(s,x_{n-1}(s))\,\mathrm{d}s+\Big\langle\int_0^t g(s,x_{n-1}(s))\,\mathrm{d}B(s)\Big\rangle,\\
S&=\int_0^t f(s,x(s))\,\mathrm{d}s+\Big\langle\int_0^t g(s,x(s))\,\mathrm{d}B(s)\Big\rangle.
\end{aligned}$$

The first term on the right-hand side of the inequality converges uniformly to zero, whereas the second is equal to zero. So it is enough to consider the third one above. By Lemma 2.9, Proposition 3.2, Proposition 3.6 and assumptions we have

$$\begin{aligned}
\delta_2^2(S_{n-1},S)\ &\le\ 2\delta_2^2\Big(\int_0^t f(s,x_{n-1}(s))\,\mathrm{d}s,\int_0^t f(s,x(s))\,\mathrm{d}s\Big)\\
&\quad+2\delta_2^2\Big(\Big\langle\int_0^t g(s,x_{n-1}(s))\,\mathrm{d}B(s)\Big\rangle,\Big\langle\int_0^t g(s,x(s))\,\mathrm{d}B(s)\Big\rangle\Big)\\
&\le\ 2L^2(t+1)\int_0^t\delta_2^2\big(x_{n-1}(s),x(s)\big)\,\mathrm{d}s\\
&\le\ 2L^2(T+1)T\sup_{t\in[0,T]}\delta_2^2\big(x_{n-1}(t),x(t)\big)\to 0,\quad\text{as}\ \ n\to\infty.
\end{aligned}$$

Therefore

$$\delta_2\Big(x(t),x_0+\int_0^t f(s,x(s))\,\mathrm{d}s+\Big\langle\int_0^t g(s,x(s))\,\mathrm{d}B(s)\Big\rangle\Big)=0\ \ \text{for every}\ t\in[0,T].$$

Hence the existence of the solution is proved. For the uniqueness assume that $x\colon[0,T]\times\Omega\to\mathcal{F}(\mathbb{R}^d)$ and $y\colon[0,T]\times\Omega\to\mathcal{F}(\mathbb{R}^d)$ are two solutions to (5). Then let us notice that

$$\delta_2^2\big(x(t),y(t)\big)\le 2L^2(T+1)\int_0^t\delta_2^2\big(x(s),y(s)\big)\,\mathrm{d}s.$$

Thus, by Gronwall's lemma, we obtain $\delta_2^2\big(x(t),y(t)\big)\le 0$ for every $t\in[0,T]$. This implies that for every $t\in[0,T]$ it holds

$$P\big([x(t)]^\alpha=[y(t)]^\alpha,\ \ \forall\alpha\in[0,1]\big)=1,$$

what ends the proof. □

Now we want to indicate that some results from a classical crisp stochastic differential equations theory are a part of the approach proposed in this paper. Indeed, let us consider a crisp stochastic differential equation

$$\mathrm{d}y(t) = a(t, y(t))\,\mathrm{d}t + b(s, y(s))\,\mathrm{d}B(s), \quad y(0) = y_0, \tag{7}$$

where B is a Brownian motion as earlier, $y_0 \colon \Omega \to I\!R^d$ is a square integrable $I\!R^d$-valued random variable which is $\mathcal{A}_0$-measurable. Let the coefficients $a, b \colon [0,T] \times \Omega \times I\!R^d \to I\!R^d$ satisfy:

— $a(\cdot,\cdot,r), b(\cdot,\cdot,r) \colon [0,T] \times \Omega \to I\!R^d$ are the nonanticipating, $I\!R^d$-valued stochastic processes, for every $r \in I\!R^d$,

— there exists a constant $L > 0$ such that P-a.e. for every $t \in [0,T]$, every $r_1, r_2 \in I\!R^d$

$$\max\{\|a(t,r_1) - a(t,r_2)\|, \|b(t,r_1) - b(t,r_2)\|\} \le L\|r_1 - r_2\|,$$

— there exists a constant $C > 0$ such that P-a.e. for every $(t,r) \in [0,T] \times I\!R^d$

$$\max\{\|a(t,r)\|, \|b(t,r)\|\} \le C(1 + \|r\|).$$

It is a classical result that in such a setting there exists a solution $y \colon [0,T] \times \Omega \to I\!R^d$ to (7), which is $\{\mathcal{A}_t\}$-adapted $I\!R^d$-valued square integrable stochastic process such that for every $t \in [0,T]$

$$y(t) = y_0 + \int_0^t a(s, y(s))\,\mathrm{d}s + \int_0^t b(s, y(s))\,\mathrm{d}B(s) \quad P\text{-a.e.}$$

Moreover, if $y, z \colon [0,T] \times \Omega \to I\!R^d$ are any two solutions to (7) then $P\big(y(t) = z(t)\big) = 1$ for every $t \in [0,T]$.

Let $\big\langle I\!R^d \big\rangle$ denote the image of $I\!R^d$ by the embedding $\langle \cdot \rangle \colon I\!R^d \to \mathcal{F}(I\!R^d)$.

Consider now equation (5), where $x_0 = \big\langle y_0 \big\rangle$, $f \colon [0,T] \times \Omega \times \big\langle I\!R^d \big\rangle \to \mathcal{F}(I\!R^d)$ is defined by

$$f(t,u) = \big\langle a(t,r) \big\rangle, \text{ if } t \in [0,T] \quad \text{and} \quad u = \big\langle r \big\rangle, r \in I\!R^d,$$

and $g \colon [0,T] \times \Omega \times \big\langle I\!R^d \big\rangle \to I\!R^d$ is defined by

$$g(t,u) = b(t,r), \text{ if } t \in [0,T] \quad \text{and} \quad u = \big\langle r \big\rangle, r \in I\!R^d.$$

It is a matter of simple calculations to check that x_0, f, g satisfy assumptions of Theorem 4.4. Hence a unique solution x to (5) exists. It is clear that $x = \big\langle y \big\rangle$, where y is the solution to the crisp problem (7).

Example 4.5. Let us take a fuzzy random variable $x_0\colon \Omega \to \mathcal{F}(I\!R)$ as $x_0 = \langle y_0 \rangle$, where $y_0\colon \Omega \to I\!R$ is a crisp random variable such that $I\!E|y_0|^2 < \infty$. Let $f\colon [0,T] \times \Omega \times \langle I\!R \rangle \to \mathcal{F}(I\!R)$, $g\colon [0,T] \times \Omega \times \langle I\!R \rangle \to I\!R$ be as follows

$$f(t,u) = \langle ar \rangle, \text{ if } t \in [0,T] \quad \text{and} \quad u = \langle r \rangle, r \in I\!R,$$

$$g(t,u) = br, \text{ if } t \in [0,T] \quad \text{and} \quad u = \langle r \rangle, r \in I\!R,$$

where $a, b \in I\!R \setminus \{0\}$. Then due to Theorem 4.4 the equation (5), for f, g, x_0 as above, has a unique solution $x\colon [0,T] \times \Omega \to \mathcal{F}(I\!R)$. Moreover, for this solution x we have

$$x(t) = \big\langle y_0 \exp\big\{(a - b^2/2)t + bB_t\big\}\big\rangle \quad \text{for } t \in [0,T].$$

The next result presents the boundedness of the solution to (5).

Theorem 4.6. Let $x_0 \in \mathcal{L}^2(\Omega, \mathcal{A}, P; \mathcal{F}(I\!R^d))$ and let $f\colon [0,T] \times \Omega \times \mathcal{F}(I\!R^d) \to \mathcal{F}(I\!R^d)$, $g\colon [0,T] \times \Omega \times \mathcal{F}(I\!R^d) \to I\!R^d$ satisfy the assumptions of Theorem 4.4. Then the solution x to the equation (5) satisfies

$$\sup_{t \in [0,T]} \delta_2^2\big(x(t), \hat{\theta}\big) \le 3\big(\delta_2^2\big(x_0, \hat{\theta}\big) + 2C^2T(T+1)\big)e^{6C^2T(T+1)}.$$

P r o o f. Since for every $t \in [0,T]$

$$\delta_2^2\big(x(t), \hat{\theta}\big) = \delta_2^2\Big(x_0 + \int_0^t f(s, x(s))\,\mathrm{d}s + \Big\langle \int_0^t g(s, x(s))\,\mathrm{d}B(s) \Big\rangle, \hat{\theta}\Big),$$

using Lemma 2.9, Proposition 3.2 and Proposition 3.6 we can write the following estimation for $\delta_2^2\big(x(t), \hat{\theta}\big)$:

$$\delta_2^2\big(x(t), \hat{\theta}\big) \le 3\delta_2^2(x_0, \hat{\theta}) + 3T \int_0^t \delta_2^2\big(f(s, x(s)), \hat{\theta}\big)\,\mathrm{d}s + 3\int_0^t \delta_2^2\big(\langle g(s, x(s))\rangle, \hat{\theta}\big)\,\mathrm{d}s.$$

By assumption (c3) we obtain

$$\delta_2^2\big(x(t), \hat{\theta}\big) \le 3\delta_2^2\big(x_0, \hat{\theta}\big) + 6C^2T(T+1) + 6C^2(T+1)\int_0^t \delta_2^2\big(x(s), \hat{\theta}\big)\,\mathrm{d}s.$$

Hence, by Gronwall's lemma, we get the assertion. □

In the sequel we want to give some estimation for the distance of the solutions of the two fuzzy stochastic differential equations. In what follows let $y_0, z_0 \in \mathcal{L}^2(\Omega, \mathcal{A}, P; \mathcal{F}(I\!R^d))$, $f_1, f_2\colon [0,T] \times \Omega \times \mathcal{F}(I\!R^d) \to \mathcal{F}(I\!R^d)$, $g_1, g_2\colon [0,T] \times \Omega \times \mathcal{F}(I\!R^d) \to I\!R^d$ satisfy the same assumptions as x_0 and f, g in Theorem 4.4, respectively. Let us denote by y, z the solutions to the stochastic fuzzy differential equations written in their symbolic form:

$$\mathrm{d}y(t) = f_1(t, y(t))\,\mathrm{d}t + \big\langle g_1(t, y(t))\,\mathrm{d}B(t)\big\rangle, \quad y(0) = y_0, \tag{8}$$

$$\mathrm{d}z(t) = (f_1 + f_2)(t, z(t))\,\mathrm{d}t + \big\langle (g_1 + g_2)(t, z(t))\,\mathrm{d}B(t)\big\rangle, \quad z(0) = z_0, \tag{9}$$

respectively, where $(f_1 + f_2)(t, \omega, u) = f_1(t, \omega, u) + f_2(t, \omega, u)$ for every $(t, \omega, u) \in [0, T] \times \Omega \times \mathcal{F}(\mathbb{R}^d)$.

Theorem 4.7. Assume that $y, z\colon [0, T] \times \Omega \to \mathcal{F}(\mathbb{R}^d)$ are the solutions to the problems (8), (9), respectively. Then

(i) the following inequality holds true

$$\sup_{t\in[0,T]} \delta_2^2(y(t), z(t)) \le \Big[3\delta_2^2(y_0, z_0) + 12C^2T(T+1)\big(1 + \sup_{t\in[0,T]} \delta_2^2\big(z(t), \hat{\theta}\big)\big)\Big] e^{6L^2T(T+1)},$$

(ii) if there exists a constant $K \ge 0$ such that for every $(t, u) \in [0, T] \times \mathcal{F}(\mathbb{R}^d)$ it holds

$$\max\Big\{\delta_2\big(f_2(t, u), \hat{\theta}\big), \big(\mathbb{E}\|g_2(t, u)\|^2\big)^{1/2}\Big\} \le K,$$

then

$$\sup_{t\in[0,T]} \delta_2^2\big(y(t), z(t)\big) \le \big(3\delta_2^2(y_0, z_0) + 6T(T+1)K^2\big)\, e^{6L^2T(T+1)}.$$

P r o o f. We shall prove (i). Notice that for every $t \in [0, T]$

$$\begin{aligned}\delta_2^2\big(y(t), z(t)\big) \le\ & 3\delta_2^2(y_0, z_0) \\ & + 6T \int_0^t \Big(\delta_2^2\big(f_1(s, y(s)), f_1(s, z(s))\big) + \delta_2^2\big(f_2(s, z(s)), \hat{\theta}\big)\Big)\,\mathrm{d}s \\ & + 6 \int_0^t \Big(\delta_2^2\big(\langle g_1(s, y(s))\rangle, \langle g_1(s, z(s))\rangle\big) + \delta_2^2\big(\langle g_2(s, z(s))\rangle, \hat{\theta}\big)\Big)\,\mathrm{d}s.\end{aligned}$$

Now the result follows when we use assumptions (c2), (c3) and the Gronwall lemma. The proof of (ii) is analogous. □

Corollary 4.8. Let the assumptions of Theorem 4.7 be satisfied. Suppose that $f_2 \equiv \hat{\theta}$, $g_2 \equiv 0$. Then

$$\sup_{t\in[0,T]} \delta_2^2\big(y(t), z(t)\big) \le 3\delta_2^2(y_0, z_0) e^{3L^2T(T+1)}.$$

Hence, it follows a continuous dependence on initial conditions of solutions to the stochastic fuzzy differential equation (5).

Finally we present a stability property of solutions to the system of stochastic fuzzy differential equations.
Let us consider the following problems:

$$\mathrm{d}x(t) = f(t, x(t))\,\mathrm{d}t + \big\langle g(t, x(t))\,\mathrm{d}B(t)\big\rangle, \quad x(0) = x_0,$$

and for $n = 1, 2, \ldots$

$$\mathrm{d}x_n(t) = f_n(t, x_n(t))\,\mathrm{d}t + \big\langle g_n(t, x_n(t))\,\mathrm{d}B(t)\big\rangle, \quad x_n(0) = x_{0,n}.$$

Theorem 4.9. Let f, g and f_n, g_n satisfy the assumptions of Theorem 4.4, i.e. the conditions (c1) − (c3) with the same constants L, C. Let also $x_0, x_{0,n}$ be such as in Theorem 4.4. If $\delta_2\big(x_{0,n}, x_0\big) \to 0$, $\delta_2\big(f_n(t, u), f(t, u)\big) \to 0$ and $\mathbb{E}\|g_n(t, u) - g(t, u)\|^2 \to 0$, for every $(t, u) \in [0, T] \times \mathcal{F}(\mathbb{R}^d)$, as $n \to \infty$, then

$$\sup_{t\in[0,T]} \delta_2\big(x_n(t), x(t)\big) \to 0, \quad \text{as } n \to \infty.$$

P r o o f. By virtue of Lemma 2.9, Proposition 3.2 and Proposition 3.6 let us note that for every $t \in [0, T]$

$$\begin{aligned}
\delta_2^2\big(x_n(t), x(t)\big) \quad &\le \quad 3\delta_2^2\big(x_{0,n}, x_0\big) + 3T \int_0^t \delta_2^2\big(f_n(s, x_n(s)), f(s, x(s))\big)\,\mathrm{d}s \\
&\quad + 3 \int_0^t \delta_2^2\big(\langle g_n(s, x_n(s))\rangle, \langle g(s, x(s))\rangle\big)\,\mathrm{d}s \\
&\le \quad 3\delta_2^2\big(x_{0,n}, x_0\big) + 6T \int_0^t \delta_2^2\big(f_n(s, x(s)), f(s, x(s))\big)\,\mathrm{d}s \\
&\quad + 6 \int_0^t \delta_2^2\big(\langle g_n(s, x(s))\rangle, \langle g(s, x(s))\rangle\big)\,\mathrm{d}s \\
&\quad + 6L^2(T + 1) \int_0^t \delta_2^2\big(x_n(s), x(s)\big)\,\mathrm{d}s.
\end{aligned}$$

Thus by Gronwall's lemma we infer that

$$\begin{aligned}
\delta_2^2\big(x_n(t), x(t)\big) \quad &\le \quad \Big(3\delta_2^2\big(x_{0,n}, x_0\big) + 6T \int_0^t \delta_2^2\big(f_n(s, x(s)), f(s, x(s))\big)\,\mathrm{d}s \\
&\quad + 6 \int_0^t \delta_2^2\big(\langle g_n(s, x(s))\rangle, \langle g(s, x(s))\rangle\big)\,\mathrm{d}s\Big) e^{6L^2T(T+1)}.
\end{aligned}$$

Hence, by the assumptions and the Lebesgue dominated convergence theorem, the proof is completed. □

5. APPLICATION TO A MODEL OF POPULATION DYNAMICS

Consider a population of some species, which lives on a given territory. Let $x(t)$ denote the number of individuals in the underlying population at the instant t. A classical, crisp, deterministic model of the evolution of given population is described by the Malthus differential equation:

$$x'(t) = (r - m)x(t), \quad x(0) = x_0, \tag{10}$$

where r, m are the constants which describe a reproduction coefficient and mortality coefficient, respectively. The symbol x_0 denotes the initial number of individuals. The solution x of this equation is: $x(t) = x_0 \exp\{at\}$, where $a = r - m$. Assume further that $a \neq 0$. Let us recall that with the equation (10) one can associate an equivalent integral equation: $x(t) = x_0 + a \int_0^t x(s)\,\mathrm{d}s$.

In the sequel we shall transform the preceding model to the case, when some uncertainties in $x(t)$ appear. Let us introduce an observer (who watches this population) to the considerations. Assume that the state of the population depends on random factors, and that the observer can describe the state of the population only in linguistics, i. e. he is able to say that the population is, for example, "very small", "small", "not big", "big", "large" etc. In this way we incorporate two types of uncertainty to the population growth model. The first kind of uncertainty locates in *Probability Theory*, while the second is well suited to *Fuzzy Set Theory.* At this stage we could write the model with uncertainties as:

$$x(t,\omega) = x_0(\omega) + \int_0^t ax(s,\omega)\,\mathrm{d}s, \tag{11}$$

where ω symbolizes a random factor (a probability space $(\Omega, \mathcal{A}, P)$ is considered, $\omega \in \Omega$), x_0 is a fuzzy random variable, the integral is now a fuzzy integral, and the solution x is now a fuzzy stochastic process $x\colon [0,T]\times\Omega \to \mathcal{F}(\mathbb{R})$. Such problem (11) has its differential counterpart, and exemplifies the random fuzzy integral equations or, equivalently, random fuzzy differential equations (see [13]).

Assume further that some individuals emigrate from their territory and the alien individuals immigrate to the population, and this happens in very chaotic manner. Let the aggregated immigration process be modelled by the Brownian motion B. Now the population dynamics could be modelled by the equation involving uncertainties:

$$x(t,\omega) = x_0(\omega) + \int_0^t ax(s,\omega)\,\mathrm{d}s + \langle B(t,\omega)\rangle.$$

This equation can be rewritten as (in the sequel we do not write the argument ω):

$$x(t) = x_0 + \int_0^t ax(s)\,\mathrm{d}s + \Big\langle \int_0^t \mathrm{d}B(s) \Big\rangle, \tag{12}$$

or in symbolic, differential form as:

$$\mathrm{d}x(t) = ax(t)\,\mathrm{d}t + \big\langle\, \mathrm{d}B(t)\big\rangle, \quad x(0) = x_0. \tag{13}$$

So we arrived to the stochastic fuzzy differential equation of type (5), where $f\colon [0,T]\times \Omega \times \mathcal{F}(I\!R) \to \mathcal{F}(I\!R)$ is defined by $f(t,u) = a \cdot u$, and $g\colon [0,T] \times \Omega \times \mathcal{F}(I\!R) \to I\!R$ is defined by $g(t,u) \equiv 1$. Such the equation coefficients satisfy conditions (c1)–(c3). So assuming that $x_0\colon \Omega \to \mathcal{F}(I\!R)$ is a fuzzy random variable such that $x_0 \in \mathcal{L}^2(\Omega, \mathcal{A}, P; \mathcal{F}(I\!R))$ and x_0 is $\mathcal{A}_0$-measurable, the equation (13), or equivalently equation (12), has a unique solution.

In the sequel we shall establish the explicit solution to (12) with $a \neq 0$. To this end let us denote the α-levels ($\alpha \in [0,1]$) of the solution $x\colon [0,T] \times \Omega \to \mathcal{F}(I\!R)$ and α-levels of initial value $x_0\colon \Omega \to \mathcal{F}(I\!R)$ as

$$[x(t)]^\alpha = [L_\alpha(t), U_\alpha(t)] \quad \text{and} \quad [x_0]^\alpha = [x^\alpha_{0,L}, x^\alpha_{0,U}],$$

respectively. Obviously, $L_\alpha, U_\alpha\colon [0,T] \times \Omega \to I\!R$ are the stochastic processes, also $x^\alpha_{0,L}, x^\alpha_{0,U}\colon \Omega \to I\!R$ are the random variables. If the fuzzy stochastic process x is a solution to (12), then for every $t \in [0,T]$ the following property should hold

$$P\Big([x(t)]^\alpha = [x_0]^\alpha + \Big[\int_0^t ax(s)\,\mathrm{d}s\Big]^\alpha + \Big[\Big\langle \int_0^t \mathrm{d}B(s) \Big\rangle\Big]^\alpha,\ \forall \alpha \in [0,1]\Big) = 1.$$

Hence we are interested in solving the following systems of crisp stochastic integral equations:
for $a > 0$

$$\begin{cases} L_\alpha(t) &= x^\alpha_{0,L} + a\int\limits_0^t L_\alpha(s)ds + \int\limits_0^t \mathrm{d}B(s), \\ U_\alpha(t) &= x^\alpha_{0,U} + a\int\limits_0^t U_\alpha(s)ds + \int\limits_0^t \mathrm{d}B(s), \end{cases} \tag{14}$$

and for $a < 0$

$$\begin{cases} L_\alpha(t) &= x^\alpha_{0,L} + a\int\limits_0^t U_\alpha(s)ds + \int\limits_0^t \mathrm{d}B(s), \\ U_\alpha(t) &= x^\alpha_{0,U} + a\int\limits_0^t L_\alpha(s)ds + \int\limits_0^t \mathrm{d}B(s). \end{cases} \tag{15}$$

Applying the Itô formula to the equations in (14) we obtain

$$L_\alpha(t) = e^{at}\Big(x^\alpha_{0,L} + \int_0^t e^{-as}\,\mathrm{d}B(s)\Big) \quad \text{and} \quad U_\alpha(t) = e^{at}\Big(x^\alpha_{0,U} + \int_0^t e^{-as}\,\mathrm{d}B(s)\Big),$$

which implies that the solution $x\colon [0,T] \times \Omega \to \mathcal{F}(I\!R)$ to (12) with $a > 0$ is of the form

$$x(t) = e^{at} \cdot \Big(x_0 + \Big\langle \int_0^t e^{-as}\,\mathrm{d}B(s) \Big\rangle\Big).$$

To find a solution to (15) we use the classical method of fundamental matrix which applies to the systems of linear stochastic differential equations, and we obtain

$$L_\alpha(t) = \cosh(at)x^\alpha_{0,L} + \sinh(at)x^\alpha_{0,U} + e^{at}\int_0^t e^{-as}\,\mathrm{d}B(s)$$

and

$$U_\alpha(t) = \sinh(at)x^\alpha_{0,L} + \cosh(at)x^\alpha_{0,U} + e^{at}\int_0^t e^{-as}\,\mathrm{d}B(s).$$

Hence the solution $x\colon [0,T]\times\Omega \to \mathcal{F}(I\!R)$ to (12) with $a < 0$ should have the α-levels as above, i. e.

$$x(t) = \cosh(at)\cdot x_0 + \sinh(at)\cdot x_0 + \left\langle e^{at}\int_0^t e^{-as}\,\mathrm{d}B(s)\right\rangle.$$

Since for $a < 0$ and $t \in (0,T]$ the expressions $\cosh(at), \sinh(at)$ are of the opposite sign, one cannot rewrite the above solution in the form of solution which was established in the case $a > 0$.

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