ABSTRACTS

13th International Conference on Fibonacci Numbers and Their Applications

The Edouard Lucas Invited Lecture: Carl Pomerance

Primality testing—variations on a theme of Lucas

One of the earliest applications of Fibonacci numbers and more general linear recurrent sequences was in deciding if certain large numbers are prime. The fundamental idea was introduced by Lucas in the nineteenth century and developed by Lehmer and his school in the middle part of the twentieth century. It is perhaps not so appreciated that many of their thoughts persist to the current day, including in the recent deterministic, polynomial-time primality test of Agrawal, Kayal, and Saxena. This talk will survey the field of primality testing through the lens of recurrent sequences.

Talks in Alphabetical Order by Author:

ABRATE, Marco

Fundamental Theorem For Generalized Quaternions

Let $\frac{u,v}{\mathbb{F}}$ be a generalized quaternion algebra over an arbitrary field $\mathbb{F}$, that is a four dimensional vector space over F with the four basis elements $1, i_1, i_2, i_3$ satisfying the following multiplication laws:

$$i_1^2 = u, i_2^2 = v, i_3 = i_1 i_2 = -i_2 i_1,$$

and 1 acting as the unit element.

We first show the existence of a recurring relation on the powers of elements in $(\frac{u,v}{\mathbb{F}})$, and then we show how Dickson polynomials of both first and second kind can be used to derive explicit formulas for computing the zeros of the polynomials of the form $P(x) = x^n - q$, where q lies in a generalized quaternion algebra over an arbitrary field $\mathbb{F}$ of characteristic not 2.

ANDERSON, Peter G. and Ryan H. Lewis

Convolutions Associated with Tilings of the Second Kind

We consider the number of total tiles to tile an n-board using "tiling of the second kind," e.g., tiles of any length 2 or greater, or tiles of any odd length. We also investigate the number of tiles of a given length for all n-board tilings. We compare these results to earlier ones for "first kind tilings," e.g., tilings with squares and dominoes.

ANDERSON, Peter G.and Ryan H. Lewis

Board Tilings of the Second Kind

n-board tilings of the second kind involve an unlimited number of tile lengths and, possibly, several colors. Counting these yields new combinatorial interpretations of many linear recurrences.

BACCHELLI, Silvia, Luca Ferrariy and Renzo Pinzaniy

Mixed succession rules: the commutative case

We begin a systematic study of the enumerative combinatorics of mixed succession rules, which are succession rules such that, in the associated generating tree, the nodes are allowed to produce their sons at several different levels according to different production rules. Here we deal with a specific case, namely that of two different production rules whose rule operators commute. In this situation, we are able to give a general formula expressing the sequence associated with the mixed succession rule. Such a sequence is expressed in terms of the sequences associated with the component production rules. More precisely, let Ω_a and Σ_b be the following two succession rules:

$$\Omega_a : \left\{ \begin{array}{l} (a) \\ (k) \to (e_1(k)) \cdots (e_k(k)) \end{array} \right. , \Sigma_b : \left\{ \begin{array}{l} (b) \\ (k) \to (d_1(k)) \cdots (d_k(k)) \end{array} \right. . \tag{1}$$

Denote by $(c)\Omega_a^{+}\Sigma_b^{+2}$ the *doubled mixed succession rule associated with the pair* $\Omega_a\Sigma_b$ *with axiom* (c). This rule defines the generating tree having axiom (c) and such that each of its nodes lying at level n produces k sons at level $n+1$ according to the rule Ω_a and k sons at level $n+2$ according to the rule Σ_b. If the two rule operators (see [FP]) of Ω_a and Σ_b commute, then we can prove the following result.

Theorem 1 *Denote by* $\mu_r^{(s)}(x) = \Sigma_i \mu_{r,i}^{(s)} x^i$ *the* r^{th} *level polynomial of the generating tree of* Σ_s *(this means, by definition, that* $\mu_{r,i}^{(s)}$ *is the number of nodes labelled* i *at level* r *of the generating tree of* Σ_s*)). Let* $(l_n^{(a)})_{n\in N}$ *be the numerical sequence associated with* Ω_a *(i.e. the sequence counting the number of nodes at each level of the generating tree). If* $(f_n^{(c)})_{n\in N}$ *is the sequence determined by* $(c)\Omega_a^{+1}\Sigma_b^{+2}$*, we have:*

$$f_n^{(c)} = \sum_{k\geq 0} \binom{n-k}{k} \sum_i \mu_{k,i}^{(c)} l_{n-k}^{(i)} \tag{2}$$

We also provide many examples illustrating the above theorem, and several sequences of [Sl] arise in this way.

BALLOT, Christian

The $1/3 - 2/3$ *prime proportion in* $\alpha^n + \bar{\alpha}^n$

It seems that most companion Lucas sequences $\alpha^n + \bar{\alpha}^n$ have a two-third density of prime divisors. In this talk, we discuss whether this statement may be given a rigorous basis, before presenting some simple heuristics that account for this phenomenon.

BARBERO, Stefano and Umberto Cerruti

Operators acting on sequences

We studied particular operators acting on sequences $a = \{a_n\}_{n=0}^{+\infty}, a_n \in R$ where R is a ring and a_0 is an invertible element:

Interpolated Invert operator $I^{(x)}$

$I^{(x)}(a) = \{P_n(x)\}_{n=0}^{+\infty}$ where $I^{(x)}(a)$ has generating function

$$P(t) = \sum_{n=0}^{+\infty} P_n(x)t^n = \frac{\sum_{n=0}^{+\infty} a_n t^n}{1 - xt\sum_{n=0}^{+\infty} a_n t^n}$$

Interpolated Binomial operator $L^{(y)}$

$$L^{(y)}(a) = \left\{ l_n = \sum_{j=0}^{n} \binom{n}{j} y^{n-j} a_j \right\}_{n=0}^{+\infty}$$

Revert operator η

$\eta(a) = b$ where $b = \{b_n\}_{n=0}^{+\infty}$ is the inverse sequence related to a in this way

$$\begin{cases} u = u(t) = \sum_{n=0}^{+\infty} a_n t^{n+1} \\ t = t(u) = \sum_{n=0}^{+\infty} b_n u^{n+1} \end{cases}$$

We show their composition generates a group,whose action on linear recurring sequences of order two has some interesting properties. In particular we find some relations involving Motzkin paths moments,Catalan numbers and Dickson polynomials of second kind.Finally we explore some connections with Nottingham group,Bell polynomials, Dickson polynomials of second kind and Riordan group.

BENJAMIN,Arthur (with Larry Ericksen, Pallavi Jayawant, and Mark Shattuck.)

"Combinatorial Trigonometry through Chebyshev Polynomials"

Using a tiling model for Fibonacci numbers and Chebyshev polynomials, we give an original combinatorial proof that $cos(nx) = T_n(cosx)$, where T_n is the n^{th} Chebyshev polynomial. This is joint work with Larry Ericksen, Pallavi Jayawant, and Mark Shattuck.

BICKNELL-JOHNSON, Marjorie and Colin Paul Spears

Lucas Quotient Lemmas

In current pertaining to asymmetric cell division and recursive phyllotaxic patterning in biologic structures, data was organized in rectangular tables with a Fibonacci number of columns. Analysis of data arising from the division of one positive Fibonacci number by another gave a surprising relationship to Lucas numbers: quotients that rounded off to Lucas numbers. That the remainders are Fibonacci numbers was known but the almost-Lucas quotients in the lemmas following seem to be new.

Lucas Quotient Lemma 1: When F_p is divided by $F_m, 3m > p \geq 0$, the quotient "rounds off" (either up or down) to a Lucas number. The remainder is a Fibonacci number or its negative.

Lucas Quotient Lemma 2: When a Lucas number L_p is divided by $L_m, 3m > p > m > 0$, the quotient "rounds off" to Lucas number. The (non-zero) remainder is either a Lucas number or its negative.

BODAS, Medha and Samih Obaid

Generalizations of the Bramhagupta Polynomials

We introduce two generalizations of the Bramhagupta matrix and deduce several new Bramhagupta sequences of polynomials which contain most of the known Bramhagupta sequences including the Fibonacci, Lucas, Pell and Chebyshev sequences of polynomials. We also obtain two generalizations of Simson's formula which was introduced in 1753 and we find several generating functions for the new sequences. In addition we introduce new recurrence relations, several identities and new summations involving these sequences of polynomials.

CERIN, Zvonko and Zagrebu, Hrvatska

Triangles with coordinates of vertices from Pell and Pell-Lucas number

In this joint paper with Gian Mario Gianella, we consider triangles in the plane with coordinates of points from the Pell and the Pell-Lucas sequences. It is possible to take for both coordinates consecutive either Pell numbers or Pell-Lucas numbers or mix these two kinds of numbers taking for the first coordinates Pell numbers and for the second coordinates Pell-Lucas numbers and vice verse. For these four infinite sequences of triangles we explore what geometric properties they share or how are they related to each other. We also calculate some of their quantities like area, Brocard angles, and distances of certain central points when these are rather simple expressions of Pell and Pell-Lucas numbers. Sometimes, these results give interesting relations among Pell and Pell-Lucas numbers.

COOK, Charles K. and Rebecca A. Hillman

On Products of Fibonacci Numbers and Their Recurrence Relations

Various products of Fibonacci numbers and their generalizations are investigated and recurrence relations for these products are obtained.

COOPER, Curtis

Lucas $(a_1, a_2, \cdots, a_k = 1)$ *Sequences and Pseudoprimes*

Bisht defined a generalized Lucas integral sequence of order $k \geq 1$ as

$$G_n = x_1^n + x_2^n + \cdots + x_k^n,$$

where x_1, x_2, …, x_k are the roots of the equation

$$x^k = a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_k$$

with integral coefficients and $a_k \neq 0$. He proved that these sequences satisfy the congruence

$$G_p \equiv G_1 \pmod{p}$$

when p is prime. Imposing the condition $a_k = 1$, we extend these generalized Lucas integral sequence to negative indices and define these sequences as Lucas $(a_1, a_2, \ldots, a_k = 1)$ sequences.

We then prove that

$$G_{-p} \equiv G_{-1} \pmod{p}$$

when p is prime. Finally, we define the concept of Lucas $(a_1, a_2, \ldots, a_k = 1)$ pseudoprime and study some particular examples, including Perrin pseudoprimes.

DAFNIS, Spiros, D. Frosso, S. Makri and Andreas N. Philippou

Restricted Occupancy of S Kinds of Cells and Generalized Pascal Triangles

There are several well known formulas counting the number of distinct allocations of n indistinguishable objects into m distinguishable cells, each of which has capacity $k-1$. In the present paper we generalize four of them by relaxing the assumption that each of the m cells has capacity $k-1$ and assuming instead that there are s kinds of cells and each cell of kind i has capacity $k_i - 1 (i = 1, \cdots, s)$. A generalization of the Pascal triangles of order k is also discussed.

DILCHER, Karl

Stern polynomials and continued fractions

We derive new identities for a polynomial analogue of the Stern sequence and define two subsequences of these polynomials. We obtain various properties for these two interrelated

DRAZIOTIS, Konstantinos A.

On the Ljunggren Equation $y^2 = 2x^4 - 1$

We study the Ljunggren diophantine equation $y^2 = 2x^4 - 1$. The solutions are $(|x|, |y|) = (1, 1), (13, 239)$. The first proof was given by Ljunggren in [3]. Since the proof was quite complicated, Mordell asked if one could find a simpler proof. In [4] Tzanakis and Steiner gave a proof using the theory of Baker. Another proof was given by Chen [1], using the Thue-Siegel method combined with Pade approximation on algebraic functions. Also a third proof is given by the author in [2], reducing the problem to the study of a unit equation in a quartic number field. In this talk we shall describe this method.

ERICKSEN, Larry

Fibonacci Representations And Wythoff Arrays

As the sum of Fibonacci numbers using minimal terms, the Zeckendorf representations of the natural numbers are reviewed along with their unique placement in a Wythoff array. We then restrict allowable representations to Fibonacci numbers F_i with indices having the same residue j, defined by congruences $i \equiv j \pmod{m}$ for a given modulus m. And we modify the Zeckendorf generating algorithm to permit multiples of the allowed Fibonacci numbers to be used in the representations.

For binary cases of modulus 2, we collect the minimal representations into two Wythoff arrays, depending on whether the column parity is the same or opposite the index parity of the Fibonacci representations. And we present algorithms to generate the elements of the different Wythoff arrays. For every natural number, these two Wythoff arrays identify the minimal representations unique to that parity constraint. Likewise in the case for modulus 3, we give a minimal Fibonacci representation for every natural number and associate those representations with a Wythoff array.

We examine the patterns inherent in the indexation and multiplicity of the Fibonacci representations. And we give the unique infinite sequences corresponding to representation characteristics, as in the count of different Fibonacci numbers or in the total count of all Fibonacci numbers for the representation.

FAGIOLINI, Adriano (with Aldo Balestrino and Giancarlo Zini)

Generalized Fibonacci Dynamical Systems

In this paper we consider generalizations of dynamical systems that are based on the Fibonacci sequence. We first study stability properties of such systems for both the continuous- and discrete-time case. Then, by considering the Kronecker operator, we introduce a further class of dynamical systems whose outputs can be used to define possible generalization of the golden section. Applications of such system may range from realization of digital filters, manufacturing of tissue with fractal property, etc. Properties of sequences generated by these systems are partially considered and has to be further addressed.

GLEZ-REGUERAL, Ramon

An Entry Point Algorithm for High-Speed Factorization

Factorization of large integers plays an essential role in computer science and applied mathematics. A method is described which characterizes an efficient procedure for higher-speed factorization of large numbers. The algorithm for direct computing of the entry point for any natural number in the Fibonacci sequence is the keystone that completes a comprehensive theory and application of a new principle. Advantage is taken here from special Fibonaccian properties. The method applies a good ordering approach that significantly reduces the number of prime numbers to be tested by the existing trial-and-error methods. The factorized Fibonacci sequence is so used as a relational database where each Fn file contains the complete set of prime factors for a given number. A fully automated computer procedure can be applied to construct such a database. This technique opens new ways for mathematical studies using it as a computing tool for research in areas covering from theory of numbers through cryptography.

HARBORTH, Heiko with Jens-P. Bode.

"Independent Chess Pieces on Fibonacci Boards"

Corresponding to chessboards there are Fibonacci boards with triangles or heptagons as cells. For the chess-like pieces grid and king the independence number is discussed.

HILLMAN, Rebecca A., Michael R. Bacon and Charles K. Cook

Triangular Number Patterns in the Coefficients and Diagonal Sequences of Zernike and Related Polynomials

Triangular number patterns are found to exist in the coefficients of the radial polynomials of Zernike and those of Bhatia-Wolf. Diagonal sequences of these polynomial coefficients are also found to exhibit triangular number patterns. Generalizations of both sets of these polynomials, which have their foundations in optics, are also investigated for triangular number patterns.

IWAMOTO, Seiichi and Akifumi Kira

Fibonaci Complementary Duality in Optimization

We consider a pair of primal and dual quadric optimization problems

$$(P)\quad \begin{array}{ll} \text{minimize} & \sum_{k=0}^{n}[x_k^2 + (x_k - x_{k+1})^2] \\ \text{subject to} & (i)\quad x \in R^{n+2} \\ & (ii)\quad x_{n+1} = c \end{array}$$

and

$$(D)\quad \begin{array}{ll} \text{Mximize} & -\mu_0^2 - \sum_{k=0}^{n-1}[\mu_k^2 + (\mu_k - \mu_{k+1})^2] - \mu_n^2 + 2c\mu_n \\ \text{subject to} & (i)\quad \mu \in R^{n+1} \end{array}$$

where $c \in R^1$. It is shown that both optimal solutions are characterized by the Fibonacci sequence in the following sense. (i) The value of minimum and maximum are the same (duality). It is a quadratic function of c, whose coefficient is ratio of adjacent Fibonacci sequences. (ii) The minimum point and the maximum point are two-step Fibonacci sequences (Fibonacci). (iii) Both the optimum points constitute alternately the (one-step) Fibonacci sequence (complement). This triplet is called *Fibonacci complementary duality.*

IWAMOTO, Seiichi

The Golden optimum solution in quadratic programming

This paper considers a two-variable minimum/maximum distance problem and its related problems. We show that the quadratic optimization problem has an optimal solution which constructs the Golden rectangle. We call this solution the Golden optimum solution. We associates the distance (main) problem with an inverse problem, whose optimal solution constructs the Golden rectangle, too. It is shown that an inverse relation which preserves the Golden optimality holds between main and inverse problems. Further we accompany each quadratic problem with the Golden optimal solution both in terms of the Golden ratio ϕ and in terms of $\sqrt{5}$.

KALMAN, Liptai

"Diophantine equations and balancing numbers"

A. Behera, G. K. Panda in the Fibonacci Quarterly in 1999 studied balancing numbers. A positive integer n is called a balancing number if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

for some $r \in \mathbf{N}$. We investigate a generalization of balancing numbers. In our case let y, k, l be fixed positive integers with $y > 1$. We call the positive integer x, $(x \leq y)$, (k,l)-balancing number for y if

$$1^k + 2^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

KIMBERLING, Clark

Beatty Sequences Generated Several Ways

Beatty sequences are often defined in terms of irrational numbers, but here they are generated by "self-defining" recurrences. Other rules for generating these sequences, stated in terms of fractional parts, are also presented. Many of the results can be viewed as generalizations of a particular pair of Beatty sequences – the lower and upper Wythoff sequences – for which the irrational numbers are the golden ratio and its square.

KIMBERLING, Clark and Peter Moses

Complementary Equations and Zeckendorf Arrays

A solution of the complementary equation $b(n) = a(a(a(n))) + 1$, or $b = a^3 + 1$, is the pair of sequences $a = (1, 3, 4, 5, 7, ...)$ and $b = (2, 6, 8, 11, 15, ...)$. The sequence b-1 is the first column of the 3-Zeckendorf array, Z, and b is also the ordered union of columns of Z numbered $3k-1fork = 1, 2, 3, ...$ The sequences a and b also satisfy the complementary equations $b(n) = a^2 + n$, and $ab = a + b$, and $ba = a + b - 1$. It is conjectured that the general equation $b = a^m + 1$ is solved using columns of the m-Zeckendorf array. Experimental results are included.

LEE, Moon Ho and Veselin Vl. Vavrek

Hadamard Circulants and Recurrences

There is an open conjecture, which states, that there is not exists a circulant Hadamard matrices of order bigger than 4. In this paper we shall give an idea to attack this problem using recurrence relations. Shortly the idea can be explained as follow. Any two rows a and b of Hadamard matrices satisfy $\sum a_i/b_i = 0$. Write these equations for circulant matrix, and eliminate the denominators then we can obtain a polynomial equations, invariant under the action of dihedral group. Next we consider the matrix, obtained as follow: Get arbitrary diagonal matrix and replace an elements right above main diagonal with $+1$, and right below with -1. Determinant of this matrix can be calculated recursively, while the determinant of its slide modification gives as a system of equations, which are also invariant under the action of dihedral group. The idea is that probably we can use classical invariant theory, to find a connection between both systems of equations.

LEE, Moon Ho and Veselin Vl. Vavrek

Fibonacci Jacket Conference Matrices

Using recurrence sequences we construct some special type of multi-cyclic matrices, which can be easily converted to complex Hadamard Conference matrices. Next we obtain a regular Conference matrices, and show that they are equivalent to well known Paley Conference matrices.

LUCA, Florian and F. Nicolae

"$\phi(F_n) = F_m$"

Let $\phi(m)$ be the Euler function of the positive integer m. In my talk, I will indicate the main ideas of the proof of the fact that if

$$\phi(F_m) = F_n$$

then $m = 1, 2, 3, 4$. This proof uses sieves, a result of McIntosh and Roettger concerning the validity of Wall's conjecture for primes $p \leq 10^{14}$, and several computations.

MATYAS, Ferenc

On the generalization of the Fibonacci-coefficient polynomials

In the lecture the zeros of polynomials defined recursively are investigated, where the coefficients of these polynomials are the terms of a given second order linear recursive sequence of integers ($R_0 = 0, R_1 = 1, R_n = AR_{n-1} + R_{n-2}$ if $n \geq 2$). Some results on the Fibonacci-coefficient polynomials (when $A = B = 1$) obtained by D. Garth, D. Mills and P. Mitchell will be generalized.

MONGOVEN, Casey

A Style of Music Characterized by Fibonacci Numbers and the Golden Ratio

Many composers of the 20th century used Fibonacci numbers and the golden ratio in their works. None of these composers, however, made them the basis of a style. Creating a style of music characterized by Fibonacci numbers and the golden ratio requires discarding musical traditions and rethinking stylistic elements from the ground up. In this style, mathematical properties of sequences related to the golden ratio and Fibonacci numbers are converted into musical properties. This paper introduces the style, presenting three short works based on Fibonacci-related sequences.

MUNAGI, Augustine O.

"Generalized Alternating Subsets with Permutations"

In this talk we present new proofs of theorems on alternating subsets of integers by means of bijective transformations. We show that practically all known results are consequences of a simple lemma on the residue class of an integer. We also extend the notion of alternating subset to permutations of $\{1, 2, \ldots, n\}$, for the first time, and obtain the solutions to the problems of Terquem and Skolem's generalization for permutations.

MUNAGI, Augustine O.

Alternating Sequences and a Theorem of Carlitz

Carlitz (*Discrete Math.* 17 (1977), 133 - 138) generalized the notion of alternating sequence as follows. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ denote a sequence of positive integers. An increasing sequence $(x_1, x_2, \ldots, x_k)$, $k \geq r$, is said to be α-alternating if the first α_1 elements have the same parity, the next α_2 have opposite parity, the next α_3 elements have the parity of the first, and so on. This talk will discuss a simpler derivation of Carlitz's formula for the number $f(\alpha, n, k)$ of α-alternating k-subsets of $\{1, 2, \ldots, n\}$. In addition a recurrence relation for $f(\alpha, n, k)$ will be presented for the first time.

OHTSUKA, Hideyuki and Shigeru Nakamura

On some new Fibonacci identities

The first author has found some new Fibonacci and Lucas identities. One of them is as follows:

THEOREM:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = F_{n-2} - \delta$$

where $\delta \in \{0, 1\}$ *and* $n \equiv \delta \pmod 2$.

The first author has found some new sum formulae for the reciprocal Fibonacci numbers. Two of them are as follows:

THEOREM 1: $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & (n : even) \\ F_{n-2} - 1 & (n : odd) \end{cases}$$

THEOREM 2: $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}F_{n+1} & (n : even) \\ F_{n-2}F_{n+1} - 1 & (n : odd) \end{cases}$$

In this note, we shall prove these theorems.

OLAJOS, Péter

About (a, b)-type Balancing Numbers

Let $a, b, n \in N$ and $(a, b) = 1$. A positive $an + b \in N$ is called (a, b)*-type balancing number* if

$$(a + b) + (2a + b) + \cdots + (a(n-1) + b) = (a(n+1) + b) + \cdots + (a(n+r) + b)$$

for some $r \in N$. We prove interesting properties of these numbers. In the case of special values of a and b we prove special recurrence relations between balancing-numbers. As applications we provide several examples.

OLLERTON, R. L.

Counting i-paths

An i-path consists of i non-intersecting paths on a lattice using only vertical and horizontal moves, beginning (LHS) on i points on a descending diagonal and ending (RHS) on i consecutive points on a descending diagonal. i-paths arise in a number of physical contexts including electronic circuits. This paper derives recurrence and other functional relationships for the number of i-paths of length n which lead naturally to binomial, Catalan, Baxter and higher order Hoggatt numbers and related triangles described previously by Fielder & Alford.

OMUR, Nese and Yucel Turker Ulutas

On Fibonacci k-Vectors and Their Applications

Fibonacci vector geometry is the study of properties of vectors whose coordinates are drawn from integer sequences which are generated by linear recurrence equations. In 1994, John Turner developed an important ideas and themes in Fibonacci vector geometry and in 2006, Sergio Falcon and Angel Plaza defined the Fibonacci k-numbers. In this work, Fibonacci k-vectors related with Fibonacci k-numbers are defined, some elementary geometric results concerning the geometry of the triangle ABC, and Fibonacci k-vectors identities are studied, where vectors are 3-dimensional with integer coordinates.

OZEKI, Kiyota

On Melhams sum

In this paper we discuss Melham's sum, $L_1 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1}$, and inverse relation of the Chebysev type. We give explicit expressions for Melham's sum as a polynomial of F_{2n+1}.

PETHE, Dr. S. P.

Fibonacci Sequence and Its Extensions Through Differential Equation

Let $G_n = G_{n-1} + G_{n-2} + p(n)$ (1) with $G_0 = G_1 = 1$ be a difference equation. In a paper in 1989 [1], Peter R. J. Asveld considered a differential equation $x''(t) + x'(t) - x(t) = p(t)$ (2) with $x(0) = c, x'(0) = d$ corresponding to (1) and expressed the solution of (2) in terms of the Fibonacci Numbers F_n. We use Asveld's method to express the solution of $y''(t) + py'(t) - qy(t) = r(t)$ with $y(0) = a, y'(0) = b$ in terms of generalized Fibonacci sequence. By choosing various values of a and b we arrive at its various other generalizations as established earlier in [2], [3] and [4]. We show that various identities for these sequences can be established in a simpler way.

SHANNON, A.G. and C.K. Wong

Some Properties of Generalized Third Order Pell Numbers

This paper considers properties of the third order recursive sequence defined by the linear recurrence relation,

$$u_{m.n} = 2^m u_{m,n-2} + u_{m,n-3},\ n \leq 3$$

with appropriate initial conditions. The present work follows on from the case $m = 0$ (Shannon et al). Relationships with the well-known sequences of Fibonacci, Lucas and Pell are developed. The motivation for the study was to find analogous results to some of the second order classic identities such as, for example, Simson's identity

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}, n \geq 0,$$

which becomes

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{n+1} = \begin{bmatrix} u_{0,n-2} & u_{0,n} & u_{0,n-1} \\ u_{0,n-1} & u_{0,n+1} & u_{0,n} \\ u_{0,n} & u_{0,n+2} & u_{0,n+1} \end{bmatrix}, n \geq 0$$

and, the Pythagorean triple

$$(u_{2,n}u_{3,n+3})^2 + (8u_{2,n+1}u_{2,n+3}^2 = (u_{2,n}^2 + 8u_{2,n+1}u_{2,n+3})^2$$

by analogy with Horadam's Fibonacci number triple

$$(f_n f_{n+3})^2 + (2f_{n+1}f_{n+2})^2 = (f_{2,n}^2 + 2f_{n+1}f_{n+2})^2.$$

SHONHIWA, Temba

On Compositeness in Multicompositions

The paper investigates the structure of the highly composite multicomposition function $g_m(n)$ which enumerates the number of m-compositions of n into relatively prime positive summands. Among other results, we generalize a known result by establishing that the

$$lcm\Big(m(m+2), (m+1)^{\frac{n}{p_1 p_2 \cdots p_r}-1}, (m+1)^{\phi(p_1^{\alpha_1})} - 1, (m+1)^{\phi(p_2^{\alpha_2})} - 1, \cdots, (m+1)^{\phi(p_r^{\alpha_r})} - 1\Big)$$

divides $g_m(n); \forall n \geq 3$, where $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ and $lcm(a, b)$ denotes the least common multiple of a and b.

SOMER, Lawrence

On Recurrences over Algebraic Number Fields Containing A d^{th} Root of Unity

By a theorem of Wall, if $m \geq 3$ is an integer, then the period of the Fibonacci sequence modulo m is divisible by 2. We generalize this theorem to certain k^{th}- order recurrences over the ring of integers of an algebraic number filed containing a d^{th} root of unity.

SPEARS, Colin Paul, Marjorie Bicknell-Johnson and John J. Yan

Fibonacci Phyllotaxis by Asymmetric Cell Division: Zeckendorf and Wythoff Trees

This paper reports on a *Matlab* program that represents asymmetric cell division and generates the n^{th} row of the Fibonacci tree. Asymmetric cell division with a lag by newborn cells before continuous division and with lateral self-association in one dimension can be represented over unit cell-cycle time by classic Fibonacci trees. Both Wythoff and Zeckendorf forms of the classic Fibonacci tree are explored for identifiers of Horizontal Para-Fibonacci (HPF, cell Age), Zeckendorf (Z, cell generation), and Vertical Para-Fibonacci (VPF) cousinship sequences [15: A0335612, A007895, A003603] as well as Wythoff pairs for modeling two- and three-dimensional displays. Routines were written to evaluate displays up to $F_{25} = 75,025$ and higher.

Rectangular and helical displays of F_n populations parsed Fm demonstrate regular Fibonacci phyllotaxis and floret formation with uniform self-association by Age. Generation Z clusters occur with the Age motif as potential centers of nodal growth. Sequence VPF relates successive sets of newborn cells by sister and first cousin relationships. The resulting patterns can be mined for explanations of the appearance of Fibonacci numbers in plant morphogenesis, with broadening of patterns to include linear streaks and symmetric groupings.

STANICA, Pante, E. Kilic and G.N. Stanica

Spectral Properties of Some Combinatorial Matrices

Let the sequence $v_n = av_{n-1} + bv_{n-2}$, with $v_0 = 2, v_1 = a$ and the matrix $H_n(v_k, b^k)$ whose (i,j)–entries are $v_k^{i+j-n-1} \left(-(-b)^k\right)^{n-j} \binom{i-1}{n-j}$. In this talk we present some results on the spectra and related questions for H_n and other combinatorial matrices, generalizing work by Carlitz, Cooper and Kennedy.

STOCKMEYER, Paul K., Lunnon, Fred, and Victor Mascolo

New Variations on the Tower of Hanoi

The Tower of Hanoi puzzle, invented by Edouard Lucas in 1883, is well known to all students of discrete mathematics and computer science. Many variations have been proposed as exercises and challenges over the past 125 years, including some with more than three pegs that remain unsolved.

In this paper we pose several new variations, all involving two or more stacks of disks, identical except for color. The goal in each variation is to move each stack of disks from its initial location to its final location. As usual, disks must be moved one at a time, and a disk can never sit above a disk of equal or smaller diameter, regardless of color.

Hints are provided, along with solutions.

SZALAY, László , H. Belbachir and F. Bencherif

Log-concavity and unimodality in Pascal triangles

Unimodal and log-concave sequences occur in several branches of mathematics. Our main interest is to examine combinatorial sequences connected to Pascal triangle and its generalizations. A real sequence $\{a_k\}_{k=0}^{\infty}$ is *unimodal* if there exist a non-negative integer λ such that the subsequence $\{a_k\}_{k=0}^{\lambda}$ increases, while $\{a_k\}_{k=\lambda}^{\infty}$ decreases. A non-negative real sequence $\{a_k\}$ is called *logarithmically concave* (*log-concave* for short) if

$$a_k^2 \geq a_{k-1}a_{k+1}$$

for all $k \geq 1$. The log-concavity implies unimodality if the sequence has no internal zeros. After sketching historical background several new results are presented. For instance, we could show that any sequence of binomial coefficients located along a ray is log-concave.

TATAR, Gulfer, Nese Omur,and Yucel Turker Ulutas

The (p,q)- Fibonacci Hyperbolic Functions and Their Properties

The Metalic Means Family (MMF) was found by Vera W. de Spinadel in 1998. Members of MMF have the property of carrying the name of a metal among other common characteristics. Like the very well-known Golden Mean and its relatives, the Silver Mean, the Bronze Mean, the Copper Mean, the Nickel Mean and many others. The Hyperbolic Fibonacci and Lucas functions were introduced by Stakhov A.P. and Tkachenkol I. S. in 1993, and Sergio Falcon and Angel Plaza defined the k- Fibonacci Hyperbolic functions in 2006. In this paper, we define (p, q)-Fibonacci Hyperbolic functions using from roots of algebraic equations $x^2 - px - q = 0$, where p and q are natural numbers. Several properties of these (p, q)-Fibonacci Hyperbolic functions are studied. Some identities belong to Fibonacci numbers are valid in (p, q)- Fibonacci Hyperbolic functions, for example Catalan's identity, recurrence relation, d'Ocagne's identity, etc. We give an analysis of some curves and surfaces naturally related with the (p, q)- Fibonacci Hyperbolic functions.

TURNER, J. C.

Word Recurrences, with Word Reversals Radex and APFP Integer-Word Sequences

This paper is a study of a class of sequences of number-words (i.e. words whose letters are numbers) which are generated by operations on and between the number-words and letters (the letters are natural numbers).

Different means of achieving the infinite recurrence sequences are defined. The $RADEX$ operator is first used. $RADEX$ is an acronym for 'Reverse, Add, and Extend'. It operates on a single integer-word. Later another operator, denoted by $APFP$ ('Arithmetic Progression, Fibonacci Progression') is introduced. This one operates on two number-words, and produces the same sequences but with useful storage efficiencies and other advantages.

Algebraic methods are developed for studying $RADEX$ and $APFP$ sequences. Various algebraic identities are presented. Some properties of example sequences are discovered and proved. Incidences of prime numbers in the sequences are studied briefly and mainly empirically.

The first examples treated are closely related to level-sets on Schaake's Regular Knot Tree (the RKT). His tree is directly related to the well-known Stern-Brocot tree. Indeed, the *raison d'être* for this paper was to study algebraically a well-known sequence which appears on these trees, namely:

$A007305 \equiv 1, 2, 3, 3, 4, 5, 5, 4, 5, 7, 8, 7, 7, 8, 7, 5, 6, 9, 11, 10, 11, 13, 12, 9, \cdots$

YÜREKLI, Osman

Digital Roots, Vedic Multiplications and Fibonacci Numbers

While teaching a course called Multicultural Approaches to Mathematics, I became acquainted with the digital root function, its connection with the Vedic multiplication tables and visual representations of numbers. Extending these ideas to Fibonacci numbers and other well known integer sequences give rise to interesting geometrical designs. The geometrical designs are obtained using the computer algebra system Mathematica and the spread sheet program Excel.

YOUNG, Paul

A p-adic formula for the Norlund numbers and for Bernoulli numbers of the second kind

We give formulas expressing the Norlund numbers and the Bernoulli numbers of the second kind as p-adically convergent sums of traces of algebraic integers. In the case $p = 2$ we use these formulas to prove and explain the conjectures of Adelberg concerning the initial 2-adic digits of these numbers. For odd primes p we use these formulas to derive new congruences for these sequences, including a version of Kummer's congruences.

WEBB, William A.

Proving Indentities Involving Products of Recurrence Sequences And Binomial Coefficients

We show that sums of terms which are products of a binomial coefficient and a recurrence sequence are representable in a closed form as a recurrence sequence. An algorithm is described which calculates the closed form of such a sum and thus reduces the proof of any corresponding identity to a routine calculation.

WEBB, William, Bala Krishnamoorthy and Nathan Moyer

Knapsack Cryptography Using Recurrence Sequences

The traditional knapsack code is based on the binary representation of a number and an associated superincreasing sequence. The resulting codes have a low density making them susceptible to attacks using basis reduction algorithms in integer lattices. Using representations of a number as a sum of terms in a recurrence sequence, such as the Zeckendorf representation, can generate higher density codes. We examine whether such codes are more resistant to basis reduction attacks.

WITULA, Roman and Damian Slota

Central Trinomial Coefficients and Convolution Type Identities

A direct reason for the creation of this paper were classes with the students of applied physics at the Faculty of Mathematics and Physics, Silesian University of Technology in Gliwice, and strictly speaking, the undertaken attempts to expand the standard methods of calculating definite Riemann integrals (the theories of residues and functional equations) with certain combinatoric methods connected with the utilization of orthogonality of the system: $e^{inx}, n\mathbb{Z}$ in $L^2_{\mathbb{C}}([0, 2\pi])$. The attempts have resulted in a number of new ideas and combinatoric identities, inter alia supplements and enriches H. Prodingers paper (Knuths old sum a survey, EATCS Bull. 57 (1994), 232245). We provide generalization for the earlier found identities and first of all, we point out their connection with central trinomial coefficients and Legendre polynomials. In the last section, we provide many convolution-type identities, including the classical generalizing Vandermondes convolution formulas.

WITULA, Roman and Damian Slota

Quasi-Fibonacci Numbers of Order 13 on the Occasion the Thirteenth International Conference on Fibonacci Numbers and Their Applications

In this paper we introduce and investigate the so-called quasi-Fibonacci numbers of order 13. These numbers are defined by six conjugate recurrence equations of order six. We study some relationships, identities and applications concerning these numbers. For example we present some applications to the decomposition of some polynomials. Many of the identities presented here are the generalizations of the identities characteristic for general recurrence sequences of order three given by Rabinowitz (Algorithmic manipulation of third-order linear recurrences, Fib. Quart. 34 (1996), 447464). Witula and colleagues analyzed the relationships between the so-called quasi-Fibonacci numbers of orders seven and eleven. In the course of the analysis, it turned out that our quasi-Fibonacci numbers of orders seven, eleven and thirteen, despite many obvious comparable and compatible relations, have some characteristic qualities in the domain of each of the numbers group. Likewise, quasi-Fibonacci numbers of orders seven and thirteen have more common properties than quasi-Fibonacci numbers of order eleven.