THE NUMBER OF SPANNING TREES IN THE SQUARE OF A CYCLE

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INTRODUCTION

A classic result known as the *Matrix Tree Theorem* expresses the number of spanning trees $t(G)$ of a graph G as the value of a certain determinant. There are special graphs G for which the value of this determinant is known to be obtained from a simple formula. Herein, we prove the formula $t(\mathscr{C}_n^2) = nF_n^2$, where F_n is a Fibonacci number, and $\mathscr{C}_n^2$ is the square of the n vertex cycle $\mathscr{C}_n$ using Kirchoff's matrix free theorem [7].

In this work graphs are undirected and, unless otherwise noted, assumed to have no multiple edges or self-loops. We shall follow the terminology and notation of the book by Harary [5]. The graph that consists of exactly one cycle on all its vertices is denoted by $\mathscr{C}_n$. The square G^2 of a graph G has the same vertices of G but u and v are adjacent in G^2 whenever the distance between u and v in G does not exceed 2.

The number of spanning trees of a graph G, denoted by $t(G)$, is the total number of distinct spanning subgraphs that are trees. The problem of finding the number of spanning trees of a graph arises in a variety of applications. In particular, it is of interest in the analysis of electric networks. It was in this context that Kirchhoff [7] obtained a classic result known as the matrix tree theorem. To state the result, we introduce the following matrices. The *Kirchhoff matrix* M of n-vertex graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ matrix $[m_{ij}]$ where $m_{ij} = -1$ if v_i and v_j are adjacent, and m_{ii} equals the degree of vertex i.

KIRCHHOFF'S MATRIX TREE THEOREM

For any graph with two or more vertices, all the cofactors of M are equal, and the value of each cofactor equals $t(G)$.

Clearly, the matrix tree theorem solves the problem of finding the number of spanning trees of a graph. Furthermore, we note that this is an effective result from a computational standpoint, as their are efficient algorithms for evaluating a determinant. However, for certain special cases, it is possible to give an explicit, simple formula for the number of spanning trees. For example, it is easy to see that this number is n if G is $\mathscr{C}_n$. Also, if G is the complete graph K_n, then a classic result known as *Cayley's tree formula* states that $t(K_n) = n^{n-2}$ (see Harary [5] for a proof). Another graph of special interest is the *wheel* W_n which consists of a single cycle $\mathscr{C}_n$ having an additional

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vertex, called the *center*, joined by an edge to each vertex on the cycle. In the case of wheels, there is a fascinating connection between the number of spanning trees, Lucas numbers, and Fibonacci numbers. Many authors including Harary, O'Neil, Read, and Schwenk [6], Sedláček [12], Rebman [10], and Bedrosian [1] have obtained results regarding this connection. The classic result is due to Sedláček who showed that

$$t(W_n) = ((3+\sqrt{5})/2)^n + ((3-\sqrt{5})/2)^n - 2 \text{ for } n \geqslant 3.$$

Another simple graph, which is a variant of a cycle, is $\mathscr{C}_n^2$ the square of a cycle.

For $n \geqslant 5$, the squared cycle $\mathscr{C}_n^2$ has all its vertices of degree 4. For $n = 5$, $\mathscr{C}_5^2 = K_5$; for $n = 4$, $\mathscr{C}_4^2 = K_4$; however, the vertices of K_4 have degree 3. In the case $n \geqslant 5$, the matrix M can be permuted into a circulant matrix form. Here we are assuming that an $n \times n$ circulant matrix K is one in which each row is a one-element shift of the previous row, i.e., $k_{ij} = k_{i+1,j+1}$, where the indices are taken modulo n. Namely for $\mathscr{C}_n^2$, $m_{ii} = 4$, $m_{ij} = -1$ if $|i - j| = 1, 2, n - 1$, or $n - 2$, and $m_{ij} = 0$ otherwise. Alternatively, as M is a circulant, it could be specified by its first row $(4, -1, -1, 0, 0, \ldots, 0, -1, -1)$.

Recently, Boesch and Wang [2] conjectured, without knowledge of [8], that $t(\mathscr{C}_n^2) = nF_n^2$, F_n being the Fibonacci numbers $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. Herein, we prove that this formula is indeed correct. Clearly, by Kirchhoff's Theorem, if u_n denotes $t(\mathscr{C}_n^2)$, then u_n is the determinant of the $(n-1) \times (n-1)$ matrix V_{n-1}, where V_n is the following $k \times k$ matrix:

$$\begin{bmatrix}
4 & -1 & -1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -1 \\
-1 & 4 & -1 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\
-1 & -1 & 4 & -1 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
0 & -1 & -1 & 4 & -1 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\
\vdots & & & & & & & & & & \vdots \\
0 & \cdot & \cdot & \cdot & 0 & -1 & -1 & 4 & -1 & -1 & 0 \\
0 & \cdot & \cdot & \cdot & 0 & 0 & -1 & -1 & 4 & -1 & -1 \\
0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -1 & -1 & 4 & -1 \\
-1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -1 & -1 & 4
\end{bmatrix} = V_k.$$

For convenience of the proof, we introduce the following family of matrices, all of size $k \times k$:

A_k is the matrix obtained by deleting the first row and first column of V_{k+1}, whereas

$$B_k = \left[\begin{array}{c|c}
-1 & -1 \quad 0 \ldots 0 \\
\hline
-1 & \\
-1 & \\
0 & A_{k-1} \\
\vdots & \\
0 &
\end{array}\right], \qquad C_k = \left[\begin{array}{c|c}
-1 \quad -1 \quad 0 \ \ldots & 0 \\
\hline
 & \vdots \\
A_{k-1} & 0 \\
 & -1 \\
 & -1
\end{array}\right],$$

$$D_k = \left[\begin{array}{c|ccc} -1 & -1 & 0 \ldots & 0 \\ \hline 4 & & & \\ -1 & & & \\ 0 & & B_{k-1} & \\ \vdots & & & \\ 0 & & & \end{array}\right]$$

Let a_k, b_k, c_k, d_k, v_k be respectively the determinants of A_k, B_k, C_k, D_k, V_k. Note that $u_n = v_{n-1}$.

<u>Lemma 1</u>: $v_n = a_n - a_{n-2} + 2(-1)^n c_{n-1}$.

<u>Proof</u>: We use the following simple identity:

$$\det\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = (-1)^{n+1} a_{n1} \cdot \det\begin{bmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n} \end{bmatrix}$$

$$+ \det\begin{bmatrix} a_{11} & \cdots & & a_{1n} \\ \vdots & & & \\ a_{n-1,1} & & & \vdots \\ 0 & a_{n,2} & \cdots & a_{nn} \end{bmatrix} \tag{1}$$

Applying this to v_n, we obtain:

$$v_n = (-1)^n \det\left[\begin{array}{c|c} -1 \quad -1 \quad 0 \ldots 0 & -1 \\ \hline & 0 \\ & \vdots \\ A_{n-2} & 0 \\ & -1 \\ & -1 \end{array}\right] + \det\left[\begin{array}{c|c} 4 & -1 \quad -1 \quad 0 \ldots 0 \quad -1 \\ \hline -1 & \\ -1 & \\ 0 & A_{n-1} \\ \vdots & \\ 0 & \end{array}\right] \tag{2}$$

Now, applying the transpose version of (1) to each of the two matrices in (2), where M^t is the transpose of M, we get

$$v_n = (-1)^n c_{n-1} + (-1)^n (-1)^{n+1} a_{n-2} + (-1)^n \det C_{n-1}^t + a_n. \ \square$$

We now proceed to ascertain the recursions that a_n, b_n, c_n, and d_n satisfy.

<u>Lemma 2</u>: (i) $a_n = 4a_{n-1} + b_{n-1} - d_{n-1}$

(ii) $b_n = b_{n-1} - a_{n-1}$

(iii) $d_n = 5b_{n-2} - b_{n-3} - 5b_{n-1}$

(iv) $c_n = -c_{n-1} + 4c_{n-2} - c_{n-3} - c_{n-4}$

Proof: (i) is obtained by expanding A_n with respect to the first column.

(ii) If we expand B_n with respect to the first row, we get

$b_n = -a_{n-1} + \det(B^t_{n-1}) = -a_{n-1} + b_{n-1}$.

(iii) We expand D_n with respect to the first row:

$$d_n = -b_{n-1} + \det\left[\begin{array}{c|ccc} 4 & -1 & 0 \; \ldots & 0 \\ \hline -1 & & & \\ 0 & & A_{n-2} & \\ \vdots & & & \\ 0 & & & \end{array}\right],$$

and by expanding further with respect to the first row,

$$d_n = -b_{n-1} + 4a_{n-2} + \det\left[\begin{array}{c|cccc} -1 & -1 & -1 & 0 \; \ldots & 0 \\ \hline 0 & & & & \\ \vdots & & A_{n-3} & & \\ 0 & & & & \end{array}\right],$$

which is $d_n = -b_{n-1} + 4a_{n-2} - a_{n-3}$. Now, by using (ii) to substitute for a_{n-2} and a_{n-3}, we obtain the desired result.

(iv) We expand C_n with respect to the first row:

$$c_n = -c_{n-1} + \det\left[\begin{array}{c|ccc} 4 & -1 & 0 \; \ldots & 0 \\ \hline -1 & & & \\ -1 & & & \\ 0 & & C_{n-2} & \\ \vdots & & & \\ 0 & & & \end{array}\right]$$

$$= -c_{n-1} + 4c_{n-2} + \det\left[\begin{array}{c|ccc} -1 & -1 & 0 \; \ldots & 0 \\ \hline -1 & & & \\ 0 & & C_{n-3} & \\ \vdots & & & \\ 0 & & & \end{array}\right]$$

$$= -c_{n-1} + 4c_{n-2} - c_{n-3} + \det \begin{bmatrix} -1 & -1 & 0 \ldots 0 \\ 0 & & \\ \vdots & & C_{n-4} \\ 0 & & \end{bmatrix}$$

or $c_n = -c_{n-1} + 4c_{n-2} - c_{n-3} - c_{n-4}$ as desired. □

We now establish that the sequence $\{v_n\}$ (and thus $\{u_n\}$) satisfies the same recursion as nF_n^2. For convenience, we use the following terminology. If we have a sequence $\{x_n\}$ and a recursion

$$\lambda_k x_{n+k} + \lambda_{k-1} x_{n+k-1} + \cdots + \lambda_0 x_0 = 0,$$

then we say $\{x_n\}$ fulfills the recursion given by

$$\lambda_k E^k + \lambda_{k-1} E^{k-1} + \cdots + \lambda_0 E^0 = 0,$$

where E is the shift operator $Ex_n = x_{n+1}$, $E^0 = 1$, and $\lambda_0, \lambda_1, \ldots, \lambda_k$ are constants.

Lemma 3: The sequence $\{v_n\}$ fulfills

$$(E + 1)^2 (E^2 - 3E + 1)^2 = E^6 - 4E^5 + 10E^3 - 4E + 1 = 0.$$

Proof: By Lemma 1, $v_n = a_n - a_{n-2} + 2(-1)^n c_{n-1}$.

We shall first determine the recursion for b_n and, from this, determine a recursion for a_n. Then, by obtaining a recursion for c_n, we get a recursion for v_n.

By (ii) of Lemma 2 with $n = n + 1$, and by (iii) of Lemma 2 with $n = n - 1$, we obtain, by substitution in (i) of Lemma 2, that

$$b_n - b_{n+1} = a_n = 4a_{n-1} + b_{n-1} - 5b_{n-3} + b_{n-4} + 5b_{n-2}.$$

Now, substituting for a_{n-1} its value from (ii) of Lemma 2, we get

$$b_{n+1} - 5b_n + 5b_{n-1} + 5b_{n-2} - 5b_{n-3} + b_{n-4} = 0.$$

Hence, shifting the index so $b_{n+1} \to b_{n+5}$, we see that $\{b_n\}$ fulfills

$$p(E) = E^5 - 5E^4 + 5E^3 + 5E^2 - 5E + 1 = (E^2 - 3E + 1)^2 (E + 1) = 0.$$

Since $a_n = b_n - b_{n+1}$, $\{a_n\}$ fulfills the same recursion.

By Lemma 2, the sequence $\{c_n\}$ fulfills

$$q(E) = E^4 + E^3 - 4E^2 + E + 1 = (E - 1)^2 (E^2 + 3E + 1) = 0$$

and $(-1)^n c_n$ fulfills the recursion where E is to be replaced by $-E$. Which is

$$q(-E) = (E + 1)^2 (E^2 - 3E + 1) = 0.$$

Since

$$v_n = a_n - a_{n-2} + 2(-1)^n c_{n-1},$$

and $(E + 1)^2 (E^2 - 3E + 1)^2$ is a common multiple of $p(E)$ and $q(-E)$, v_n fulfills this recursion. □

Lemma 4: The sequence nF^2 fulfills

$$E^6 - 4E^5 + 10E^3 - 4E + 1 = 0.$$

Proof: Since

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

we obtain

$$nF_n^2 = \frac{n}{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2(-1)^n\right],$$

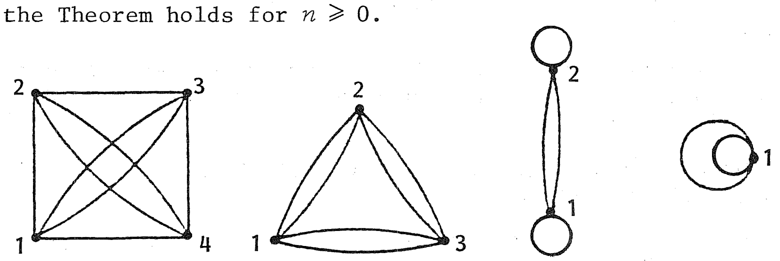
Now by the standard methods for finding the solution of a linear recursion relation via its characteristic polynomial, we see that nF_n^2 fulfills

$$\left(E - \frac{3+\sqrt{5}}{2}\right)^2 \cdot \left(E - \frac{3-\sqrt{5}}{2}\right)^2 \cdot (E+1)^2 = (E^2 - 3E + 1)^2(E+1)^2 = 0. \ \square$$

So we see that v_n, u_n, and nF_n^2 fulfill the same recursion. Since the computer computations of Boesch and Wang [2] tell us that $u_i = iF_i^2$, $5 \leqslant i \leqslant 16$, we know that the sequences coincide and have proved the following Theorem.

Theorem: The number of spanning trees of the square of the cycle $\mathscr{C}_n$, for $n \geqslant 5$, is given by nF_n^2.

Remarks: If we consider the square of a cycle for $n < 5$, which means that we consider the edge set to be a multiset, we have multiple edges and loops and the Theorem holds for $n \geqslant 0$.



$\mathscr{C}_4^2$ $4 \cdot 3^2 = 36$ | $\mathscr{C}_3^2$ $3 \cdot 2^2 = 12$ | $\mathscr{C}_2^2$ $2 \cdot 1^2 = 2$ | $\mathscr{C}_1^2$ $1 \cdot 1^2 = 1$ | $\mathscr{C}_0^2$ $0 \cdot 0^2 = 0$

Figure 1

In closing, we note that there is an alternative approcah to finding $t(\mathscr{C}_n^2)$ that uses the properties of circulant matrices. First, we note that M can be written as $4I - A$, where I is the identity matrix and A is the adjacency matrix of $\mathscr{C}_n^2$. If the maximum eigenvalue of the real, symmetric matrix A is denoted by λ_n, then a result of Sachs [11] states that

$$t(\mathscr{C}^2) = \frac{1}{n}\prod_{i=1}^{n-1}(4 - \lambda_i),$$

where λ_i are the eigenvalues of A. Now, using the explicit formulas for the eigenvalues of a circulant matrix (see, for example, Marcus and Minc [9]), one obtains

$$nt(\mathscr{C}^2) = \prod_{k=1}^{n-1} 4\sin^2\frac{\pi k}{n}\left(1 + 4\cos^2\frac{\pi k}{n}\right).$$

Thus, the Theorem could be proved by showing that the above product is n^2F^2. However, we have not found this approach to be any simpler than the one given here.

The authors would like to point out that reference [8] gives a purely combinatorial proof of our result, which was conjectured by Bedrosian in [1]. Furthermore, the paper by Kleitman and Golden was not discovered until after our paper had been refereed and accepted for publication.

REFERENCES

1. S. Bedrosian. "The Fibonacci Numbers via Trigonometric Expressions." *J. Franklin Inst. 295* (1973):175-177.
2. F. T. Boesch & J. F. Wang. "A Conjecture on the Number of Spanning Trees in the Square of a Cycle." In *Notes from New York Graph Theory Day V*, p. 16. New York: Academy of Sciences, 1982.
3. S. Chaiken. "A Combinatorial Proof of the All Minors Matrix Tree Theorem." *SIAM J. Algebraic Discrete Methods 3* (1982):319-329.
4. S. Chaiken & D. Kleitman. "Matrix Tree Theorems." *J. Combinatorial Theory Ser. A 24* (1978):377-381.
5. F. Harary. *Graph Theory*. Reading, Mass.: Addison-Wesley, 1969.
6. F. Harary, P. O'Neil, R. Read, & A. Schwenk. "The Number of Trees in a Wheel." In *Combinatorics* (*Proc. Conf. Combinatorial Math., Math. Inst.*), pp. 155-163. Oxford, 1972.
7. G. Kirchhoff. "Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird." *Ann. Phys. Chem. 72* (1847):497-508.
8. D. J. Kleitman & B. Golden. "Counting Trees in a Certain Class of Graphs." *Aner. Math. Montly* (1975), pp. 40-44.
9. M. Marcus & H. Minc. *A Survey of Matrix Theory and Matrix Inequalities*. Boston: Allyn and Bacon, 1964.
10. K. Rebman. "The Sequence 1, 5, 16, 45, 121, 320, ... in Combinatorics." *The Fibonacci Quarterly 13*, no. 1 (1975):51-55.
11. H. Sachs. "Uber selbstkomplementare Graphen." *Publ. Math. Debrecen 9* (1961):270-288.
12. J. Sedláček. "Lucas Numbers in Graph Theory." In *Mathematics* (*Geometry and Graph Theory*) (*Czech.*), pp. 111-115. Prague: University of Karlova, 1970.

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