REPRESENTATIONS FOR A SPECIAL SEQUENCE

L. CARLITZ* and RICHARD SCOVILLE
Duke University, Durham, North Carolina
and
VERNER E. HOGGATT, JR.
San Jose State University, San Jose, California

1. INTRODUCTION AND SUMMARY

Consider the sequence defined by

(1.1) $$u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = u_n + 2u_{n-1} \qquad (n \geq 1).$$

It follows at once from (1.1) that

(1.2) $$u_n = \frac{1}{3}(2^n - (-1)^n), \qquad u_n + u_{n+1} = 2^n.$$

The first few values of u_n are easily computed.

n	1	2	3	4	5	6	7	8	9	10
u_n	1	1	3	5	11	21	43	85	171	341

It is not difficult to show that the sums

(1.3) $$\sum_{i=2}^{k} \epsilon_i u_i \qquad (k = 2, 3, 4, \cdots),$$

where each $\epsilon_i = 0$ or 1, are distinct. The first few numbers in (1.3) are

$$1, 3, 4, 5, 6, 8, 9, 11, 12, 14, 15, 16, 17, 19, 20, \cdots.$$

Thus there is a sequence of "missing" numbers beginning with

(1.4) $$2, 7, 10, 13, 18, 23, 28, 31, 34, 39, \cdots.$$

In order to identify the sequence (1.4) we first define an array of positive integers R in the following way. The elements of the first row are denoted by a(n), of the second row by b(n), of the third row by c(n). Put

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$$a(1) = 1, \quad b(1) = 3, \quad c(1) = 2 .$$

Assume that the first $n - 1$ columns of R have been filled. Then $a(n)$ is the smallest integer not already appearing, while

$$b(n) = a(n) + 2n \tag{1.5}$$

and

$$c(n) = b(n) - 1 . \tag{1.6}$$

The sets $\{a(n)\}$, $\{b(n)\}$, $\{c(n)\}$ constitute a disjoint partition of the positive integers. The following table is readily constructed.

n	1	2	3	4	5	6	7	8	9	10	11	12
a	1	4	5	6	9	12	15	16	17	20	21	22
b	3	8	11	14	19	24	29	32	35	40	43	48
c	2	7	10	13	18	23	28	31	34	39	42	47

The table suggests that the numbers $c(n)$ are the "missing" numbers (1.4) and we shall prove that this is indeed the case.

Let A_k Denote the set of numbers

$$\begin{cases} N = u_{k_1} + u_{k_2} + \cdots + u_{k_r} , \\ 2 \le k = k_1 < k_2 < \cdots < k_r \end{cases} \tag{1.7}$$

and $r = 1, 2, 3, \cdots$. We shall show that

$$A_{2k+2} = ab^k a(\underset{\sim}{N}) \cup ab^k c(\underset{\sim}{N}) \qquad (k \ge 0) \tag{1.8}$$

and

$$A_{2k+1} = b^k a(\underset{\sim}{N}) \cup b^k c(\underset{\sim}{N}) \qquad (k \ge 1) , \tag{1.9}$$

where N denotes the set of positive integers.

If N is given by (1.7), we define

$$e(N) = u_{k_r-1} + u_{k_r-1} + \cdots + u_{k_r-1} . \tag{1.10}$$

Then we shall show that

$$e\bigl(a(n)\bigr) = n \tag{1.11}$$

and

$$e\bigl(b(n)\bigr) = a(n) . \tag{1.12}$$

Clearly the domain of the function $c(n)$ is restricted to $a(\underset{\sim}{N}) \cup b(\underset{\sim}{N})$. However, since, as we shall see below, $\bigl(b(n) - 2\bigr) \in a(\underset{\sim}{N})$ and

(1.13) $$e(b(n) - 2) = a(n) ,$$

it is natural to define

(1.14) $$e(c(n)) = a(n) .$$

Then e(n) is defined for all n and we show that e(n) is monotone.

The functions a, b, c satisfy various relations. In particular we have

$$a^2(n) = b(n) - 2 = a(n) + 2n - 2$$
$$ab(n) = ba(n) + 2 = 2a(n) + b(n)$$
$$ac(n) = ca(n) + 2 = 2a(n) + c(n)$$
$$cb(n) = bc(n) + 2 = 2a(n) + 3c(n) + 2 .$$

Moreover if we define

(1.15) $$d(n) = a(n) + n$$

then we have

$$da(n) = 2d(n) - 2$$
$$db(n) = 4d(n)$$
$$dc(n) = 4d(n) - 2 .$$

It follows from (1.11) and (1.12) that every positive integer N can be written in the form

(1.16) $$N = u_{k_1} + u_{k_2} + \cdots + u_{k_r} ,$$

where now

$$1 \leq k_1 < k_2 < \cdots < k_r .$$

Hence N is a "missing" number if and only if $k_1 = 1$, $k_2 = 2$.

The representation (1.16) is in general not unique. The numbers a(n) are exactly those for which, in the representation (1.7), k_1 is even. Hence in (1.66) if we assume that k_1 is odd, the representation (1.16) is unique. We accordingly call this the canonical representation of N.

Returning to (1.15), we define the complementary function d'(n) so that the sets $\{d(n)\}$, $\{d'(n)\}$ constitute a disjoint partition of the positive integers. We shall show that

(1.17) $$d(n) = 2d'(n) .$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
d'	1	3	4	5	7	9	11	12	13	15	16	17	19	20	21	23
d	2	6	8	10	14	18	22	24	26	30	32	34	38	40	42	46

.

Let $\delta(n)$ denote the number of $d(k) \le n$ and let $\delta'(n)$ denote the number of $d'(k) \le n$. We show that

$$\delta(N) = \left[\frac{N}{2}\right] - \left[\frac{N}{4}\right] + \left[\frac{N}{8}\right] - \cdots$$

$$\delta'(N) = [N] - \left[\frac{N}{2}\right] + \left[\frac{N}{4}\right] - \cdots .$$

Finally, if N has the canonical representation (1.16) we define

$$f(N) = \sum_{i=1}^{r} (-1)^{k_i} . \tag{1.18}$$

It follows that

$$a(N) = 2N + f(N) \tag{1.19}$$

and

$$d(N) = a(N) + N = \sum_{i=1}^{r} 2^{k_i} , \tag{1.20}$$

so that there is a close connection with the binary representation of an integer.

Even though there is no "natural" irrationality associated with the sequence $\{u_n\}$, it is evident from the above summary that many of the results of the previous papers of this series [2, 3, 4, 5, 6] have their counterpart in the present situation.

The material in the final two sections of the paper is not included in the above summary.

2. THE CANONICAL REPRESENTATION

As in the Introduction, we define the sequence $\{u_n\}$ by means of

$$u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = u_n + 2u_{n-1} \qquad (n \ge 1) .$$

We first prove the following.

Theorem 2.1. Every positive integer N can be written uniquely in the form

$$N = \epsilon_1 u_1 + \epsilon_2 u_2 + \cdots , \tag{2.1}$$

where the $\epsilon_i = 0$ or 1 and

$$\epsilon_1 = \cdots = \epsilon_{k-1} = 0, \qquad \epsilon_k = 1 \Rightarrow k \text{ odd} . \tag{2.2}$$

Proof. The theorem can be easily proved by induction on n as follows. Let C_{2n} consist of all sequences

$$(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2n}) \qquad (\epsilon_i = 0 \text{ or } 1)$$

satisfying (2.2). Then the map

$$(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2n}) \longrightarrow \epsilon_1 u_1 + \epsilon_2 u_2 + \cdots + \epsilon_{2n} u_{2n}$$

is $1 - 1$ and onto from C_{2n} to $[0, \cdots, u_{2n+1} - 1]$. Clearly $C_2 \longrightarrow [0,1]$. Assuming that

$$C_{2n} \longrightarrow [0, \cdots, u_{2n+1} - 1] ,$$

we see that

$$\begin{aligned} C_{2n+2} \longrightarrow & [0, \cdots, u_{2n+1} - 1] \quad [u_{2n+1}, \cdots, 2u_{2n+1} - 1] \\ & \cup [u_{2n+2} + 1, \cdots, u_{2n+1} + u_{2n+2} - 1] \\ & \cup [u_{2n+1} + u_{2n+2}, \cdots, 2u_{2n+1} + u_{2n+2} - 1] \\ & = [0, \cdots, u_{2n+3} - 1] \end{aligned}$$

since

$$2u_{2n+1} - 1 = u_{2n+2} .$$

If (2.2) is satisfied we call (2.1) the canonical representation of N.

In view of the above we have also

Theorem 2.2. If N and M are given canonically by

$$N = \sum \epsilon_i u_i, \qquad M = \sum \delta_i u_i ,$$

then

(2.3) $$N \le M \quad \text{if and only if} \quad \sum \epsilon_i 2^i \le \sum \delta_i 2^i .$$

Let N be given by (2.1) and define

(2.4) $$\phi(N) = \sum \epsilon_i 2^i .$$

Note that since

(2.5) $$u_n = \frac{1}{3}(2^n - (-1)^n) ,$$

we have

(2.6) $$N = \frac{1}{3}(\phi(N) - f(N)) ,$$

where

(2.7) $$f(N) = \epsilon(-1)^i \epsilon_i .$$

Theorem 2.3. There are exactly N numbers of the form $2^k K$, k, K odd, less than or equal to $\phi(N)$.

Proof. The N numbers of the stated form are simply

$$\phi(1), \phi(2), \cdots, \phi(N) .$$

If N is given canonically by

$$N = \epsilon_1 u_1 + \epsilon_2 u_2 + \cdots ,$$

we define

(2.8) $$a(n) = \epsilon_1 u_2 + \epsilon_3 u_3 + \cdots .$$

This is of course never canonical. Define

(2.9) $$b(n) = a(N) + 2N = \epsilon_1 u_3 + \epsilon_1 u_4 + \cdots .$$

The representation (2.9) is canonical.

Suppose ϵ_{2k+1} is the first nonzero ϵ_i in the canonical representation of N. Then, since

$$u_1 + u_2 + \cdots + u_{2k+1} = u_{2k+2} ,$$

we see that a(N) is given canonically by

(2.10) $$a(n) = u_1 + u_2 + \cdots + u_{2k+1} + 0 \cdot u_{2k+2} + \epsilon_{2k+2} u_{2k+3} + \cdots .$$

Let c(N) = b(N) - 1. Then, since

$$u_1 + u_2 + \cdots + u_{2k+2} = u_{2k+3} - 1 ,$$

c(N) is given canonically by

(2.11) $$c(N) = u_1 + u_2 + \cdots + u_{2k+2} + 0 \cdot u_{2k+3} + \epsilon_{2k+2} u_{2k+4} + \cdots .$$

We now state

Theorem 2.4. The three functions a, b, c defined above are strictly monotone and their ranges $a(\underset{\sim}{N})$, $b(\underset{\sim}{N})$, $c(\underset{\sim}{N})$ form a disjoint partition of $\underset{\sim}{N}$.

Proof. We have

(2.12) $$\phi\big(a(N) + 1\big) = 2\phi(N) + 2$$

and

(2.13) $$\phi\big(b(N)\big) = 4\phi(N) .$$

Since ϕ is 1 - 1 and monotone, it follows that a, b, c are monotone. By (2.10), $a(\underset{\sim}{N})$ consists of those N whose canonical representations begin with an odd number of 1's; $b(\underset{\sim}{N})$ of those which begin with 0; and $c(\underset{\sim}{N})$ of those which begin with an even number of 1's. Hence all numbers are accounted for.

It is now clear that the functions a, b, c defined above coincide with the a, b, c defined in the Introduction.

The following two theorems are easy corollaries of the above.

Theorem 2.5. $c(\underset{\sim}{N})$ is the set of integers that cannot be written as a sum of distinct u_i with $i \geq 2$.

Thus the $c(\underset{\sim}{N})$ are the "missing" numbers of the Introduction.

Theorem 2.6. If $K \notin c(\underset{\sim}{N})$, then K can be written uniquely as a sum of distinct u_i with $i \geq 2$.

3. RELATIONS INVOLVING a, b, AND c

We now define

$$d(N) = a(N) + N.$$

Since

$$u_k + u_{k+1} = 2^k ,$$

it follows at once from (2.4) and (2.8) that

$$d(N) = \phi(N). \tag{3.1}$$

Hence, by (2.6), we may write

$$2N = a(N) - f(N) . \tag{3.2}$$

Let d' denote the monotone function whose range is the complement of the range of d. Since the range of ϕ (that is, of d) consists of the numbers $2^k K$, with k, K both odd, it follows that the range of d' consists of the numbers $2^k K$ with k even and K odd. We have therefore

$$d(N) = 2d'(N) . \tag{3.3}$$

Thus (2.12) and (2.13) become

$$d(a + 1) = 2d + 2 \tag{3.4}$$

and

$$db = 4d , \tag{3.5}$$

respectively.

From (2.10) we obtain

$$da = 2d - 2 \tag{3.6}$$

and

$$d'a = d - 1 . \tag{3.7}$$

Theorem 3.1. We have

$$a^2(N) = b(N) - 2 = a(N) + 2N - 2 \tag{3.8}$$

$$ab(N) = ba(N) + 2 = 2a(N) + b(N) \tag{3.9}$$

$$ac(N) = ca(N) + 2 = 2a(N) + c(N) \tag{3.10}$$

$$cb(N) = bc(N) + 2 = 2a(N) + 3c(N) \tag{3.11}$$

$$da(N) = 2d(N) - 2 \tag{3.12}$$

$$db(N) = 4d(N) \tag{3.13}$$

$$dc(N) = 4d(N) - 2 . \tag{3.14}$$

Proof. The first four formulas follow from the definitions. For example if

$$N = u_{2k+1} + \epsilon_{2k+2} u_{2k+2} + \cdots ,$$

then

$$a(N) = 1\cdot u_1 + 1\cdot u_2 + \cdots + 1\cdot u_{2k+1} + \epsilon_{2k+2} u_{2k+3} + \cdots$$

and

$$\begin{aligned} a^2(N) &= 1\cdot u_2 + \cdots + 1\cdot u_{2k+2} + \epsilon_{2k+2} u_{2k+4} + \cdots \\ &= u_{2k+3} - 2 + \epsilon_{2k+2} u_{2k+4} + \cdots \\ &= b(N) - 2 . \end{aligned}$$

Formula (3.12) is the same as (3.6) while (3.13) and (3.14) follow from the formulas for ab and ac.

In view of Theorem 2.6, every

$$N \in a(\underset{\sim}{N}) \cup b(\underset{\sim}{N})$$

can be written uniquely in the form

$$N = \delta_2 u_2 + \delta_3 u_3 + \cdots \tag{3.15}$$

with $\delta_2 = 0, 1$. We define A_k as the set of N for which δ_k is the first nonzero δ_i.

Theorem 3.2. We have

$$A_{2k+2} = ab^k a(\underset{\sim}{N}) \cup ab^k c(\underset{\sim}{N}) \qquad (k \geq 0) \tag{3.16}$$

$$A_{2k+1} = b^k a(\underset{\sim}{N}) \cup b^k c(\underset{\sim}{N}) \qquad (k \geq 1) . \tag{3.17}$$

Proof. By (2.9), (2.10) and (2.11), the union

$$a(\underset{\sim}{N}) \cup c(\underset{\sim}{N})$$

consists of those K for which

$$\epsilon_1 = \epsilon_1(K) = 1 .$$

Hence, applying a, we have

$$A_2 = a^2(\underset{\sim}{N}) \cup ac(\underset{\sim}{N})$$

and, applying b,

$$A_3 = ba(\underset{\sim}{N}) \cup bc(\underset{\sim}{N}) .$$

Continuing in this way, it is clear that we obtain the stated results.

Theorem 3.2 admits of the following refinement.

Theorem 3.3. We have

$$ab^k a(\underset{\sim}{N}) = \{N \in A_{2k+2} \mid N = ab^k a(n) \equiv n \pmod 2\} \tag{3.17}$$

$$ab^k c(\underset{\sim}{N}) = \{N \in A_{2k+2} \mid N = ab^k c(n) \equiv n + 1 \pmod 2\} \tag{3.18}$$

$$b^k a(\underset{\sim}{N}) = \{N \in A_{2k+1} \mid N = b^k a(n) \equiv n \pmod 2\} \tag{3.19}$$

(3.20) $$b^k c(\underset{\sim}{N}) = \{N \in A_{2k+1} \mid N = b^k c(n) \equiv n + 1 \pmod 2\}.$$

Proof. The theorem follows from Theorem 3.2 together with the observation

(3.21) $$a(n) \equiv b(n) \equiv n, \qquad c(n) \equiv n + 1 \pmod 2.$$

Let

$$N \in a(\underset{\sim}{N}) \cup b(\underset{\sim}{N}),$$

so that (3.15) is satisfied. We define

(3.21) $$e(N) = \delta_2 u_1 + \delta_3 u_2 + \cdots.$$

Then from the definition of a and b we see that

(3.22) $$e(a(n)) = n$$

and

(3.23) $$e(b(n)) = a(n).$$

Since

$$a^2(n) = b(n) - 2 < c(n) < b(n),$$

we define

(3.24) $$e(c(n)) = a(n).$$

Thus $e(n)$ is now defined for all n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
e	1	1	1	2	3	4	4	4	5	5	5	6	6	6	7	8	9	9	9	10
n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
e	11	12	12	12	13	14	15	15	15	16	16	16	17	17	17	18	19	20	20	20

Theorem 3.4. The function e is monotone. Indeed $e(n) = e(n - 1)$ if and only if

$$n \in b(\underset{\sim}{N}) \cup c(\underset{\sim}{N}).$$

Otherwise $(n \in a(\underset{\sim}{N}))$

$$e(n) = e(n - 1) + 1.$$

Proof. We have already seen that

$$e\, b(n) = e\, c(n) = e\, c(n) - 1 = a(n).$$

Thus it remains to show that

(3.25) $$e(a(n)) = e(a(n) - 1) + 1.$$

Let

$$n = u_{2k+1} + \epsilon_{2k+2} u_{2k+2} + \cdots$$

be the canonical representation of n. Then

$$a(n) = u_{2k+2} + \epsilon_{2k+2} u_{2k+3} + \cdots .$$

Since

$$u_{2k+2} - 1 = u_2 + u_3 + u_4 + \cdots + u_{2k+1} ,$$

we get

$$a(n) - 1 = u_2 + u_3 + \cdots + u_{2k+1} + \epsilon_{2k+2} u_{2k+3} + \cdots .$$

It follows that

$$\begin{aligned} e(a(n) - 1) &= u_1 + u_2 + \cdots + u_{2k} + \epsilon_{2k+2} u_{2k+2} + \cdots \\ &= (u_{2k+1} - 1) + \epsilon_{2k+2} u_{2k+2} + \cdots \\ &= n - 1 . \end{aligned}$$

This evidently proves (3.25).

Theorem 3.5. We have

$$(3.26) \qquad \begin{cases} a(n + 1) = a(n) + 3 & (n \in a(\mathbb{N})) \\ a(n + 1) = a(n) + 1 & (n \in b(\mathbb{N}) \cup c(\mathbb{N})) . \end{cases}$$

Proof. Formula (3.4) is evidently equivalent to

$$(3.27) \qquad a(a(n) + 1) = b(n) + 1 .$$

By (3.8)

$$a^2(n) = b(n) - 2 = c(n) - 1 ,$$

so that we have the sequence of consecutive integers

$$(3.28) \qquad a^2(n), \quad c(n), \quad b(n), \quad a\ a(n) + 1 \ .$$

On the other hand, by (3.9) and (3.10)

$$ab(n) = ac(n) + 1 .$$

Finally, since

$$b(n) + 1 \in a(n) ,$$

we have, by (3.28),

$$\begin{aligned} a(b(n) + 1) &= a^2(a(n) + 1) = b\ a(n) + 1 \ - 2 \\ &= a(a(n) + 1) + 2a(n) \\ &= 2a(n) + b(n) + 1 \\ &= ab(n) + 1 \qquad . \end{aligned}$$

This completes the proof of the theorem.

If we let $\alpha(n)$ denote the number of $a(k) < n$, it follows at once from Theorem 3.5 that

$$(3.29) \qquad a(n) = n + 2\alpha(n) \qquad (n \geq 1) .$$

This is equivalent to

(3.30) $$d'(n) = n + \alpha(n) .$$

We shall now show that

(3.31) $$\alpha(n + 1) = e(n) .$$

Let $n \in a(\underset{\sim}{N}) \cup b(\underset{\sim}{N})$. Then

$$n = u_k = \epsilon_{k+1} u_{k+1} + \cdots \qquad (k \geq 2)$$

and

$$e(n) = u_{k-1} + \epsilon_{k+1} u_k + \cdots .$$

Also

$$n + 1 = u_1 + u_k + \epsilon_{k+1} u_{k+1} + \cdots \qquad \text{(canonical)} ,$$

so that

$$a(n + 1) = u_2 + u_{k+1} + \epsilon_{k+1} u_{k+2} + \cdots .$$

It follows that

(3.32) $$a(n + 1) - 2e(n) = n + 1 \qquad (n \notin c(\underset{\sim}{N})) .$$

If $n \in c(\underset{\sim}{N})$ we have $e(n) = e(n+1)$. Since $n + 1 \in b(\underset{\sim}{N})$, we may use (3.32). Thus

$$2e(n) = 2e(n + 1) = a(n + 2) - (n + 2) = a(n + 1) - (n + 1) ,$$

by (3.26). Hence

$$a(n + 1) - 2e(n) = n + 1$$

for all n. This is evidently equivalent to (3.31).

This proves

Theorem 3.6. The number of $a(k) \leq n$ is equal to $e(n)$. Moreover

(3.33) $$a(n) = n + 2e(n - 1) \qquad (n > 1) .$$

A few special values of $a(n)$ may be noted:

(3.34) $$a(2^{2k-1}) = 2^{2k} \qquad (k \geq 1)$$

(3.35) $$a(2^{2k}) = 2^{2k+1} - 2 \qquad (k \geq 1)$$

(3.36) $$a(2^{2k-1} - 2) = 2^{2k} - 4 \qquad (k > 1)$$

(3.37) $$a(2^{2k} - 2) = 2^{2k+1} - 6 \qquad (k > 2) .$$

4. COMPARISON WITH THE BINARY REPRESENTATION

If N is given in its binary representation

(4.1) $$N = \gamma_0 + \gamma_1 \cdot 2 + \gamma_2 \cdot 2^2 + \cdots ,$$

where $\gamma_i = 0$ or 1, we define

(4.2) $$\delta(N) = \gamma_0 u_0 + \gamma_1 u_1 + \gamma_2 u_2 + \cdots$$

and

(4.3) $$(N) = \sum_i \gamma_i (-1)^i .$$

Then we have

(4.4) $$\delta(d(N)) = N$$

and

(4.5) $$\chi(d(N)) = f(N) .$$

A simple computation leads to

(4.6) $$\delta(N) = \left[\frac{N}{2}\right] - \left[\frac{N}{4}\right] + \left[\frac{N}{8}\right] - \cdots .$$

Let

(4.7) $$\delta'(N) = N - \left[\frac{N}{2}\right] + \left[\frac{N}{4}\right] - \cdots$$

so that

(4.8) $$\delta(N) + \delta'(N) = N .$$

Theorem 4.1. The number of $d(k) \leq n$ is equal to $\delta(N)$. The number of $d'(k) \leq n$ is equal to $\delta'(N)$.

Proof. Since δ is monotone, we have $d(k) \leq n$ if and only if

$$k = \delta\, d(k) \leq \delta(n) .$$

Hence, in view of (4.8), the theorem is proved.

We have seen in Section 3 that if N has the canonical representation

$$N = \epsilon_1 u_1 + \epsilon_2 u_2 + \cdots$$

then

(4.9) $$a(N) - 2N = f(N) ,$$

where

$$f(N) = \sum_i (-1)^i \epsilon_i .$$

It follows that

(4.10) $$d(N) = a(N) + N = \sum_i \epsilon_i \cdot 2^i .$$

Replacing N by d(N), d'(N) in (4.9), we get

(4.11) $$a(d(N)) - 2d(N) = f(d(N))$$

and

(4.12) $$a(d'(N)) - 2d'(N) = f(d(N)) .$$

Theorem 4.2. The function f(d) takes on every even value (positive, negative or zero) infinitely often. The function f(d') takes on every odd value (positive or negative) infinitely often.

Proof. Consider the number

$$\begin{aligned} N &= u_1 + u_3 + u_5 + \cdots + u_{2k-1} \\ &= \frac{1}{3}(2^1 + 1) + \frac{1}{3}(2^3 + 1) + \cdots + \frac{1}{3}(2^{2k-1} + 1) \\ &= \frac{1}{3}\left(\frac{2}{3}(2^{2k} - 1) + k\right) . \end{aligned}$$

Clearly

$$N \equiv 2 \pmod 4 \tag{4.13}$$

if and only if

$$k \equiv 0 \pmod 4 . \tag{4.14}$$

It follows from (4.13) that $N \in d(\underset{\sim}{N})$. Also it is evident that

$$f(N) = -k, \qquad k \equiv 0 \pmod 4 . \tag{4.15}$$

In the next place the number

$$\begin{aligned} N &= u_3 + u_5 + \cdots + u_{2k+1} \\ &= \frac{1}{3}(2^3 + 1) + \frac{1}{3}(2^5 + 1) + \cdots + \frac{1}{3}(2^{2k+1} + 1) \\ &\equiv 3k \pmod 8 . \end{aligned}$$

Hence for $k \equiv 2 \pmod 4$, we have $N \equiv 2 \pmod 4$ and so as above $N \in d(\underset{\sim}{N})$. Also it is evident that (4.15) holds in this case also.

Now consider

$$\begin{aligned} N &= u_1 + u_2 + u_4 + u_6 + \cdots + u_{2k} \\ &= 1 + \frac{1}{3}(2^2 - 1) + \frac{1}{3}(2^4 - 1) + \cdots + \frac{1}{3}(2^{2k} - 1) \\ &\equiv 1 + k \pmod 4 . \end{aligned}$$

Thus for k odd, $N \in d(\underset{\sim}{N})$. Also it is clear that

$$f(N) = k - 1 .$$

This evidently proves the first half of the theorem.

To form the second half of the theorem we first take

$$N = u_1 + u_3 + u_5 + \cdots + u_{2k-1} .$$

Then

$$N \equiv k \pmod 2 .$$

Thus for k odd, $N \in d'(\underset{\sim}{N})$. Moreover

(4.16) $$f(N) = -k .$$

Next for

$$N = u_1 + u_2 + u_4 + u_6 + \cdots + u_{2k} + u_{2k+2}$$

we again have

$$N \equiv k \pmod 2 ,$$

so that $N \in d'(\underset{\sim}{N})$ for k odd. Clearly

(4.17) $$f(N) = k.$$

This completes the proof of the theorem.

As an immediate corollary of Theorem 4.2 we have

Theorem 4.3. The commutator

$$ad(N) - da(N) = fd(N) + 2$$

takes on every even value infinitely often. Also the commutator

$$ad'(N) - d'a(N) = fd'(N) + 1$$

takes on every even value infinitely often.

5. WORDS

By a word function, or briefly, word, is meant a function of the form

(5.1) $$w = a^{\alpha} b^{\beta} c^{\gamma} a^{\alpha'} b^{\beta'} c^{\gamma'} \cdots ,$$

where the exponents are arbitrary non-negative integers.

Theorem 5.1. Every word function $w(n)$ can be linearized, that is

(5.2) $$w(n) = A_w a(n) + B_w n - C_w \qquad (A_w > 0) ,$$

where A_w, B_w, C_w are independent of n. Moreover the representation (5.2) is unique.

Proof. The representation (5.2) follows from the relations

(5.3) $$\begin{cases} a^2(n) = a(n) + 2n - 2 \\ ab(n) = 2a(n) + b(n) = 3a(n) + 2n \\ ac(n) = 2a(n) + c(n) = 3a(n) + 2n - 1 . \end{cases}$$

If we assume a second representation (5.2) it follows that $a(n)$ is a linear function of n. This evidently contradicts Theorem 3.5.

Theorem 5.2. For any word w, the coefficient B_w in (5.2) is even. Hence the function d is not a word.

Proof. Repeated application of (5.3).

Remark. If we had defined words as the set of functions of the form

$$a^{\alpha} b^{\beta} c^{\gamma} d^{\delta} \cdots , \tag{5.4}$$

then, in view of Theorem 4.3, we would not be able to assert the extended form of Theorem 5.1.

Combining (5.3) with (5.2), we get the following recurrences for the coefficients A_w, B_w, C_w:

$$\begin{cases} A_{wa} = A_w + B_w \\ B_{wa} = 2A_w \\ C_{wa} = 2A_w + C_w \end{cases} \tag{5.5}$$

$$\begin{cases} A_{wb} = 3A_w + B_w \\ B_{wb} = 2A_w + 2B_w \\ C_{wb} = C_w \end{cases} \tag{5.6}$$

$$\begin{cases} A_{wc} = 3A_w + B_w \\ B_{wc} = 2A_w + 2B_w \\ C_{wc} = A_w + B_w + C_w . \end{cases} \tag{5.7}$$

In particular we find that

$$a^k(n) = u_k a(n) + 2u_{k-1} n - (u_{k+1} - 1) , \tag{5.8}$$

$$ab^k(n) = u_{2k+1} a(n) + (u_{2k+1} - 1)n , \tag{5.9}$$

$$ac^k(n) = u_{2k+1} a(n) + (u_{2k+1} - 1)n - \tfrac{1}{3}(4u_{2k} - k) , \tag{5.10}$$

$$b^k(n) = u_{2k} a(n) + (u_{2k} + 1)n , \tag{5.11}$$

$$a^k b^j(n) = u_{k+2j} a(n) + 2u_{k+2j-1} n - (u_{k+1} - 1) , \tag{5.12}$$

$$b^j a^k(n) = u_{k+2j} a(n) + 2u_{k+2j-1} n - (u_{k+2j+1} - u_{2j+1}) , \tag{5.13}$$

$$\begin{aligned} a^k b^j(n) - b^j a^k(n) &= u_{k+2j+1} - u_{k+1} - u_{2j+1} + 1 \\ &= \tfrac{2}{3}(2^k - 1)(2^{2j} - 1) . \end{aligned} \tag{5.14}$$

We shall now evaluate A_w and B_w explicitly. For w as given by (5.1) we define the weight of w by means of

(5.15) $$p = p(w) = \alpha + 2\beta + 2\gamma + \alpha' + 2\beta' + 2\gamma' + \cdots .$$

We shall show that

(5.16) $$A_w = u_p, \qquad B_w = 2u_{p-1} .$$

The proof is by induction on p. For p = 1, (5.16) obviously holds. Assume that (5.16) holds up to and including the value p. By the inductive hypothesis, (5.5), (5.6), (5.7) become

(5.17) $$\begin{cases} A_{wa} = A_p + B_p = u_p + 2u_{p-1} = u_{p+1} \\ B_{wa} = 2A_p = 2u_p \end{cases}$$

(5.18) $$\begin{cases} A_{wb} = A_{wc} = 3A_p + B_p = 3u_p + 2u_{p-1} = u_{p+2} \\ B_{wp} = B_{wc} = 2A_p + 2B_p = 2u_p + 2u_{p-1} = u_{p+1} . \end{cases}$$

This evidently completes the induction.

As for C_w, we have

(5.19) $$\begin{cases} C_{wa} = 2u_p + C_w \\ C_{wb} = C_w \\ C_{wc} = u_{p+1} + C_w . \end{cases}$$

Unlike A_w and B_w, the coefficient C_w is not a function of the weight alone. For example

$$C_{a^2} = 2, \qquad C_b = 0, \qquad C_c = 1,$$
$$C_{a^3} = 4, \qquad C_{ab} = 0, \qquad C_{ac} = 1 .$$

Repeated application of (5.19) gives

$$C_{a^k} = 2(u_1 + u_2 + \cdots + u_{k-1}) = u_{k+1} - 1$$
$$C_{b^k} = 0$$
$$C_{c^k} = u_1 + \cdots + u_k = \tfrac{1}{2}(u_{k+2} - 1) ,$$

of which the first two agree with (5.8) and (5.11).

We may state

<u>Theorem 5.3.</u> If w is a word of weight p, then

(5.20) $$w(n) - u_p a(n) + 2u_{p-1} n - C_w ,$$

where C_w can be evaluated by means of (5.19). If w, w' are any words of equal weight, then

(5.21) $$w(n) - w'(n) = C_{w'} - C_w .$$

Theorem 5.4. For any word w, the representation

$$w = a^{\alpha} b^{\beta} c^{\gamma} a^{\alpha'} b^{\beta'} c^{\gamma'} \cdots$$

is unique.

Proof. The theorem is a consequence of the following observation. If u, v are any words, then it follows from any one of

$$ua = va, \qquad ub = vb, \qquad uc = vc$$

that u = v.

Theorem 5.5. The words u, v satisfy uv = vu if and only if there is a word w such that

$$u = w^r, \qquad v = w^s ,$$

where r, s are non-negative integers.

Theorem 5.6. In the notation of Theorem 5.3, $C_w = C'_w$ if and only if $w = w'$.

Remark. It follows from (5.20) that no multiple of $d'(n)$ is a word function.

6. GENERATING FUNCTIONS

Put

$$A(x) = \sum_{n=1}^{\infty} x^{a(n)}, \qquad B(x) = \sum_{n=1}^{\infty} x^{b(n)}, \qquad C(x) = \sum_{n=1}^{\infty} x^{c(n)} \tag{6.1}$$

and

$$D(x) = \sum_{n=1}^{\infty} x^{d(n)}, \qquad D_1(x) = \sum_{n=1}^{\infty} x^{d'(n)} , \tag{6.2}$$

where of course $|x| < 1$. Then clearly

$$A(x) + B(x) + C(x) = \frac{x}{1 - x} \tag{6.3}$$

and

$$D(x) + D_1(x) = \frac{x}{1 - x}. \tag{6.4}$$

Since

$$b(n) = c(n) + 1, \qquad d(n) = 2d'(n) ,$$

(6.3) and (6.4) reduce to

$$A(x) + (1 + x)C(x) = \frac{x}{1 - x} , \tag{6.5}$$

and

$$D_1(x) + D_1(x^2) = \frac{x}{1 - x} , \tag{6.6}$$

respectively.

It follows from (6.6) that

$$D_1(x) = \frac{x}{1-x} - \frac{x^2}{1-x^2} + \frac{x^4}{1-x^4} - \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \sum_{r=1}^{\infty} x^{2^k r}$$

$$= \sum_{n=1}^{\infty} x^n \sum_{2^k r = n} (-1)^k ,$$

so that

$$d'(n) = \sum_{2^k r = n} (-1)^k .$$

This is equivalent to the result previously obtained that

$$d'(\underset{\sim}{N}) = \{2^m M \mid m \text{ even, } M \text{ odd}\}.$$

Theorem 6.1. Each of the functions $A(x)$, $B(x)$, $C(x)$, $D(x)$, $D_1(x)$ has the unit circle as a natural boundary.

Proof. It will evidently suffice to prove the theorem for $A(x)$ and $D_1(x)$. We consider first the function $D_1(x)$.

To begin with, $D_1(x)$ has a singularity at $x = 1$. Hence, by (6.6), $D_1(x)$ has a singularity at $x = -1$. Replacing x by x^2, (6.6) becomes

$$D_1(x^2) + D_1(x^4) = \frac{x^2}{1-x^2} .$$

We infer that $D_1(x)$ has singularities at $x = \pm i$. Continuing in this way we show that $D_1(x)$ has singularities at

$$x = e^{2k\pi i/2^n} \qquad (k = 1, 3, 5, \cdots, 2^n - 1; \quad n = 1, 2, 3, \cdots) .$$

This proves that $D_1(x)$ cannot be continued analytically across the unit circle.

In the next place if the function

$$f(x) = \sum_{n=1}^{\infty} c_n x^n ,$$

where the $c_n = 0$ or 1, can be continued across the unit circle, then [1, p. 315]

$$f(x) = \frac{P(x)}{1 - x^k},$$

where $P(x)$ is a polynomial and k is some positive integer. Hence

(6.7) $$c_n = c_{n-k} \qquad (n \geq n_0).$$

Now assume that $A(x)$ can be continued across the unit circle. Then by (6.7), there exists an integer k such that

$$a(n) = a(n_1) + k \qquad (n > n_0),$$

where n_1 depends on n. It follows that

(6.8) $$a(n) = a(n - r) + k \qquad (n > n_0)$$

for some fixed r. This implies

(6.9) $$d(n) = a(n - r) + k + r \qquad (n > u_0).$$

However (6.9) contradicts the fact that $D(x) = D_1(x^2)$ cannot be continued across the unit circle.

Theorem 6.2. Let $w(n)$ be an arbitrary word function of positive weight and put

(6.10) $$F_w(x) = \sum_{n=1}^{\infty} x^{w(n)}.$$

Then $F_w(x)$ cannot be continued across the unit circle.

Proof. Assume that $F_w(x)$ does admit of analytic continuation across the unit circle. Then there exist integers r, k such that

$$w(n) = w(n - r) + s \qquad (n > n_0).$$

By (5.2) this becomes

$$A_w a(n) + B_w r = A_w (n - r) + k.$$

This implies

(6.11) $$A_w d(n) = A_w d(n - r) + (A_w - B_w) r + k.$$

Since $A_w > 0$, (6.11) contradicts the fact that $D(x)$ cannot be continued.

Put

(6.12) $$E(x) = \sum_{n=1}^{\infty} x^{e(n)}.$$

Then, by Theorem 3.4,

$$E(x) = \frac{x}{1-x} + 2A(x) \,. \tag{6.13}$$

Also

$$(1 - x)^{-1}A(x) = \sum_{n=1}^{\infty} e(n)\, x^n \,. \tag{6.14}$$

In the next place, by (3.8), (3.9), and (3.10),

$$A(x) = \sum_{1}^{\infty} x^{a^2(n)} + \sum_{1}^{\infty} x^{ab(n)} + \sum_{1}^{\infty} x^{ac(n)}$$

$$= x^{-2}B(x) + (1 + x^{-1})F_{ab}(x) \,.$$

Since

$$A(x) + (1 + x^{-1})B(x) = \frac{x}{1-x} \,,$$

it follows that

$$(1 + x)^2 F_{ab}(x) = (1 + x + x^2)A(x) - \frac{x}{1-x} \,. \tag{6.15}$$

Let w, w' be two words of equal weight. Then by (5.21),

$$x^{C_w} F_w(x) = x^{C_{w'}} F_{w'}(x) \,. \tag{6.16}$$

Thus it suffices to consider the functions

$$F_{a^k}(x) \qquad (k = 1, 2, 3, \cdots) \,.$$

We have

$$F_{a^k}(x) = F_{a^{k-1}}(x) + F_{a^k b}(x) + F_{a^k c}(x) \,.$$

By (5.8)

$$\begin{aligned} a^k b(n) &= u_k ab(n) + 2u_{k-1} b(n) - (u_{k+1} - 1) \\ &= u_k(3a(n) + 2n) + 2u_{k-1}(a(n) + 2n) - (u_{k+1} - 1) \\ &= (3u_k + 2u_{k-1})a(n) + 2(u_k + 2u_{k-1})n - (u_{k+1} - 1) \\ &= u_{k+2}a(n) + 2u_{k+1}n - (u_{k+1} - 1) \\ &= a^{k+2}(n) + 2^{k+1} \end{aligned}$$

$$\begin{aligned} a^k c(n) &= u_k ac(n) + 2u_{k-1} c(n) - (u_{k+1} - 1) \\ &= u_k(3a(n) + 2n - 1) + 2u_{k-1}(a(n) + 2n - 1) - (u_{k+1} - 1) \\ &= u_{k+2}a(n) + 2u_{k+1}n - (2u_{k+1} - 1) \\ &= a^{k+2}(n) + u_{k+2} \,. \end{aligned}$$

[Continued on page 550.]