On the Comonotone Natural Extension of Marginal p-Boxes

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Abstract

The relationship between several random variables is gathered by their joint distribution. While this distribution can be easily determined by the marginals when an assumption of independence is satisfied, there are situations where the random variables are connected by some dependence structure. One such structure that arises often in practice is comonotonicity. This type of dependence refers to random variables that increase or decrease simultaneously. This paper studies the property of comonotonicity when the uncertainty about the random variables is modelled using p-boxes and the induced coherent lower probabilities. In particular, we analyse the problem of finding a comonotone lower probability with given marginal p-boxes, focusing on the existence, construction and uniqueness of such a model. Also, we prove that, under some conditions, there is a most conservative comonotone lower probability with the given marginal p-boxes, that will be called the comonotone natural extension.

Keywords: Comonotonicity, lower probabilities, p-boxes, belief functions, natural extension

1. Introduction

The joint analysis of several random variables is a task of interest in many complex scenarios, because it allows us to take into account the interactions between the different factors. Of course, under the assumption of independence, multivariate distributions are easier to handle, because it allows to decompose the joint distribution function as the product of the marginals. Without the assumption of independence, Sklar's Theorem [23] allows a similar decomposition of the joint distribution function by applying a copula [18] to the marginal distributions.

Another important dependence structure is that of comonotonicity, that corresponds to the extreme case of positive dependence: there is an increasing relationship between the marginals, so they increase or decrease simultaneously. In terms of copulas, it corresponds to applying the minimum operator to the joint distribution functions. Besides its good mathematical properties, comonotonicity has shown to be an interesting tool when dealing with risk measures [8], stochastic orderings [5, 15, 17] or finance models [6, 7], for instance.

In this paper, we assume that the probability distributions, both the marginals and the joint, cannot be elicited with total precision. In this framework, we can use the models within the Theory of Imprecise Probabilities [1, 25] to model our available information. In particular, here we consider two models: coherent lower probabilities [25] and p-boxes [9]. From the epistemic point of view, they can be used to model the available information about a real but unknown probability measure or cumulative distribution function. Even if probabilities and distribution functions are equivalent, this is not the case of coherent lower probabilities and p-boxes, the latter being less expressive. Nevertheless, p-boxes possess very good properties from the practical point of view, because they are related to belief functions [22] and their credal set can be easily computed [14].

The notion of independence has been widely investigated in the field of imprecise probabilities [2, 4, 11]. Moreover, several attempts were made in the last years to model dependence with imprecise probability models. In particular, Sklar's Theorem has been adapted when dealing with minitive belief functions [21] or random sets [20], or even when the probabilistic information is given in terms of uni- and bivariate p-boxes [16]. Also, the notion of comonotonicity for lower probabilities was given in [13], where the main properties of comonotone lower probabilities were studied.

One of the main properties of comonotone probability measures is that given two marginal probability measures, there is always a joint comonotone probability measure with the given marginals, and this joint model is unique and it can be easily computed. The problem of building a joint comonotone lower probability with given marginals was only studied in [13] for very particular cases such as possibility measures, giving some sufficient and some necessary conditions for its existence. The main aim of this paper is to investigate in depth how to build a comonotone lower probability with given marginals. More in detail, we model the uncertainty about the marginal models using p-boxes and we investigate whether we can find a comonotone lower probability with the given marginals. This comonotone model will be called a comonotone extension. In particular, we wonder whether such comonotone extension (i) always exists; (ii) can be built; (iii) is unique. In particular, we will see that, even if the comonotone extension does not always exist, it is possible to characterise its existence in terms of its marginal p-boxes. Also, when it exists we show a constructive method for building it, but we also show that, when it exists, it may not be unique. This leads us to our second aim in this paper: is it possible to characterise the least informative comonotone extension, (what we shall call comonotone natural extension)? We shall prove that, although such an extension may not exist in general, we can give necessary and sufficient conditions for its existence as well as a constructive procedure

After introducing the main notions about lower probabilities, belief functions and p-boxes in Section 2, we recall the definition of comonotonicity for probability measures and coherent lower probabilities in Section 3. Then, we analyse the existence, construction and uniqueness of a comonotone extension and the comonotone natural extension in Sections 4 and 5, respectively. We conclude the paper in Section 6 with some final comments. Note that, due to space limitations, proofs have been omitted.

2. Preliminaries

In this section we review the main tools we will use in the paper, namely lower probabilities, belief functions and (uni- and bi-variate) p-boxes.

2.1. Lower Probabilities

Consider a finite possibility space Ω and denote by $\mathbb{P}(\Omega)$ the set of all the (finitely additive) probability measures in $\mathscr{P}(\Omega)$. A *lower probability* is a function $\underline{P} : \mathscr{P}(\Omega) \to [0,1]$ satisfying the normalisation properties $\underline{P}(\emptyset) = 0$ and $\underline{P}(\Omega) = 1$ as well as monotonicity: $A \subseteq B$ implies $\underline{P}(A) \leq \underline{P}(B)$. From the lower probability we can define an *upper probability* using the conjugacy relation given by:

$$\overline{P}(A) = 1 - \underline{P}(A^c) \quad \forall A \subseteq \Omega.$$

The probabilistic information given by the lower probability can be summarised using a closed and convex set of probabilities, usually called *credal set*, given by:

$$\mathscr{M}(\underline{P}) = \left\{ P \in \mathbb{P}(\Omega) \mid P(A) \geq \underline{P}(A) \ \ \forall A \subseteq \Omega \right\}.$$

We assume that $\underline{P}$ satisfies the rationality property of *coherence*, which means that

$$\underline{P}(A) = \min \left\{ P(A) \mid P \in \mathscr{M}(\underline{P}) \right\} \quad \forall A \subseteq \Omega.$$

A coherent lower probability is completely monotone when the function $m : \mathscr{P}(\Omega) \to \mathbb{R}$ given by:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \underline{P}(B) \quad \forall A \subseteq \Omega$$

is non-negative. In that case, m is called *basic probability assignment*, and it allows to retrieve the initial lower probability using the following formula:

$$\underline{P}(A) = \sum_{B \subseteq A} m(B) \quad \forall B \subseteq \Omega. \tag{1}$$

Completely monotone lower probabilities are also called *belief functions* and play a key role in Evidence Theory [22]. Also, a belief function is characterised using the *focal events*, which are those events A with strictly positive basic assignment: $m(A) > 0$. The reason is that once that we know the focal events and their basic probability assignment, Equation (1) can be simplified, because we just need to consider the focal events included in A instead of all the subevents of A.

In this paper, we follow an epistemic interpretation of lower probabilities. This means that there is a real but unknown probability measure P_0 modelling our uncertainty. In that case, the lower probability $\underline{P}$ and its conjugate upper probability $\overline{P}$ are lower and upper bounds for P_0: $\underline{P}(A) \leq P_0(A) \leq \overline{P}(A)$ for every $A \subseteq \Omega$. Following this interpretation, $\mathscr{M}(\underline{P})$ contains the candidates for being the real but unknown probability measure P_0, meaning that we know that $P_0 \in \mathscr{M}(\underline{P})$.

2.2. Random Variables and Random Vectors

Consider now a random variable X taking values in the possibility space $\mathscr{X} = \{x_1, \ldots, x_n\}$ that is endowed with the order $x_1 < \ldots < x_n$. Our uncertainty about X can be modelled with a lower probability $\underline{P}_X$ and its conjugate upper probability $\overline{P}_X$, but instead we can also use a *probability box* (or p-box, for short). A (univariate) p-box $(\underline{F}_X, \overline{F}_X)$ [9] is a pair of cumulative distribution functions $\underline{F}_X, \overline{F}_X : \mathscr{X} \to [0,1]$ satisfying $\underline{F}_X \leq \overline{F}_X$. Following the epistemic interpretation, a p-box can be used to model the available information about the cumulative distribution function associated with P_0, F_{P_0}. From the p-box we can also define a credal set:

$$\mathscr{M}(\underline{F}_X, \overline{F}_X) = \left\{ P \in \mathbb{P}(\mathscr{X}) \mid \underline{F}_X \leq F_P \leq \overline{F}_X \right\}.$$

Also, taking lower and upper envelopes, we obtain a coherent lower probability and its conjugate upper probability:

$$\underline{P}_{(\underline{F}_X, \overline{F}_X)}(A) = \min \left\{ P(A) \mid \underline{F}_X \leq F_P \leq \overline{F}_X \right\},$$
$$\overline{P}_{(\underline{F}_X, \overline{F}_X)}(A) = \max \left\{ P(A) \mid \underline{F}_X \leq F_P \leq \overline{F}_X \right\} \quad \forall A \subseteq \mathscr{X}.$$

One important property is that the probabilistic information encoded by the p-box and its associated lower probability is the same, meaning that $\mathscr{M}(\underline{F}, \overline{F}) = \mathscr{M}\left(\underline{P}_{(\underline{F}_X, \overline{F}_X)}\right)$. Also, the lower probability $\underline{P}_{(\underline{F}_X, \overline{F}_X)}$ is not only coherent but it is also a belief function whose focal events are ordered intervals [24, Thm.17].

Consider now two random variables X and Y taking values in the possibility spaces $\mathscr{X} = \{x_1, \ldots, x_n\}$ and $\mathscr{Y} = \{y_1, \ldots, y_m\}$ satisfying $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_m$. The probabilistic information about the joint distribution can be modelled using a coherent lower probability $\underline{P}_{X,Y}$. Instead, as in the univariate framework, we can also consider a *bivariate p-box*. A bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ is a pair of functions $\underline{F}_{X,Y}, \overline{F}_{X,Y} : \mathscr{X} \times \mathscr{Y} \to [0,1]$ such that $\underline{F}_{X,Y} \leq \overline{F}_{X,Y}$, they are component-wise increasing and normalised: $\underline{F}_{X,Y}(x_n, y_m) = \overline{F}_{X,Y}(x_n, y_m) = 1$. Note that $\underline{F}_{X,Y}, \overline{F}_{X,Y}$ need not be bivariate cumulative distribution functions since they may not satisfy the rectangle inequality (see [19, Ex.1]). Again, we can consider its credal set, given by:

$$\mathscr{M}\big(\underline{F}_{X,Y}, \overline{F}_{X,Y}\big) = \big\{P \in \mathbb{P}(\mathscr{X} \times \mathscr{Y}) \mid \underline{F}_{X,Y} \leq F_P \leq \overline{F}_{X,Y}\big\}.$$

In contrast to the univariate framework, the lower and upper envelopes of this credal set, given by:

$$\underline{P}_{X,Y}(A) = \min\big\{P(A) \mid \underline{F} \leq F_P \leq \overline{F}\big\},$$
$$\overline{P}_{X,Y}(A) = \max\big\{P(A) \mid \underline{F} \leq F_P \leq \overline{F}\big\} \quad \forall A \subseteq \mathscr{X} \times \mathscr{Y},$$

may not be coherent.

Also, given a coherent lower probability $\underline{P}_{X,Y} : \mathscr{P}(\mathscr{X} \times \mathscr{Y}) \to [0,1]$ and its conjugate upper probability $\overline{P}_{X,Y}$ we can define a bivariate p-box as:

$$\underline{F}_{X,Y}(x_i, y_j) = \underline{P}_{X,Y}(\{x_1, \ldots, x_i\} \times \{y_1, \ldots, y_j\}), \quad (2)$$
$$\overline{F}_{X,Y}(x_i, y_j) = \overline{P}_{X,Y}(\{x_1, \ldots, x_i\} \times \{y_1, \ldots, y_j\}), \quad (3)$$

for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Also, from the coherent lower probability $\underline{P}_{X,Y}$ and its conjugate $\overline{P}_{X,Y}$, we can define the *marginal coherent lower probabilities* $\underline{P}_X$ and $\underline{P}_Y$, given by:

$$\underline{P}_X(A) = \underline{P}_{X,Y}(A \times \mathscr{Y}) \quad \forall A \subseteq \mathscr{X},$$
$$\underline{P}_Y(B) = \underline{P}_{X,Y}(\mathscr{X} \times B) \quad \forall B \subseteq \mathscr{Y},$$

as well as the *marginal p-boxes* $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$:

$$\underline{F}_X(x_i) = \underline{F}_{X,Y}(x_i, y_m) = \underline{P}_{X,Y}(\{x_1, \ldots, x_i\} \times \mathscr{Y}), \quad (4)$$
$$\overline{F}_X(x_i) = \overline{F}_{X,Y}(x_i, y_m) = \overline{P}_{X,Y}(\{x_1, \ldots, x_i\} \times \mathscr{Y}),$$

for every $i = 1, \ldots, n$, and

$$\underline{F}_Y(y_j) = \underline{F}_{X,Y}(x_n, y_i) = \underline{P}_{X,Y}(\mathscr{X} \times \{y_1, \ldots, y_j\}), \quad (5)$$
$$\overline{F}_Y(y_j) = \overline{F}_{X,Y}(x_n, y_i) = \overline{P}_{X,Y}(\mathscr{X} \times \{y_1, \ldots, y_j\}),$$

for every $j = 1, \ldots, m$.

From now on, we consider two univariate p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ representing our uncertainty about the marginals X and Y. For the sake of simplicity, we assume that $\overline{P}_{(\underline{F}_X, \overline{F}_X)}(\{x_i\}) > 0$ and $\overline{P}_{(\underline{F}_Y, \overline{F}_Y)}(\{y_j\}) > 0$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$. According to [24, Prop.4], this is equivalent to $\underline{F}_X(x_i) < \overline{F}_X(x_{i+1})$ and $\underline{F}_Y(y_j) < \overline{F}_Y(y_{j+1})$ for every $i = 1, \ldots, n-1$ and $j = 1, \ldots, m-1$.

3. Comonotonicity

In this paper we deal with one type of dependence structure called *comonotonicity*. In this section, we first review the definition of comonotone random variables and the equivalent representations. Later, we analyse the definition of comonotonicity given in [13] for lower probabilities.

Before starting, recall the following notation:

- Two elements $(x_i, y_j), (x_k, y_l) \in \mathscr{X} \times \mathscr{Y}$ are *comonotone* if $x_i < x_k$ implies $y_j \leq y_l$ and $y_j < y_l$ implies $x_i \leq x_k$.
- An event $A \subseteq \mathscr{X} \times \mathscr{Y}$ is *increasing* if all the elements in A are comonotone.

3.1. Comonotone Probability Measures

Given a random vector (X, Y) with joint probability $P_{X,Y}$, its support is defined as:

$$\text{Supp}(P_{X,Y}) = \big\{(x, y) \in \mathscr{X} \times \mathscr{Y} \mid P_{X,Y}\big(\{(x, y)\}\big) > 0\big\}.$$

Using the support, we can define the notion of comonotonicity.

Definition 1 *Given a random vector (X, Y), $P_{X,Y}$ is comonotone if its support $\mathit{Supp}(P_{X,Y})$ is an increasing set in $\mathscr{X} \times \mathscr{Y}$.*

According to this definition, comonotone random variables are those whose support is increasing, or in other words, when there exists an increasing relationship between them. The next theorem, that can be found for example in [6, Thm.2], [10, Sec.2] or [3, Prop.2.1], shows some equivalent representations of comonotonicity.

Theorem 2 *Given a random vector (X, Y), $P_{X,Y}$ is comonotone if and only if any, hence all, of the following conditions holds:*

1. *$\mathit{Supp}(P_{X,Y})$ is an increasing set in $\mathscr{X} \times \mathscr{Y}$.*
2. *For every (x, y), $P\big(D_1^{(x,y)}\big) = 0$ or $P\big(D_2^{(x,y)}\big) = 0$, where*
$$D_1^{(x,y)} = \{(x_i, y_j) \in \mathscr{X} \times \mathscr{Y} \mid x_i > x,\ y_j \leq y\},$$
$$D_2^{(x,y)} = \{(x_i, y_j) \in \mathscr{X} \times \mathscr{Y} \mid x_i \leq x,\ y_j > y\}.$$
3. *$F_{X,Y}(x, y) = \min\{F_X(x), F_Y(y)\}$ for every $(x, y) \in \mathscr{X} \times \mathscr{Y}$.*

This result shows some alternative equivalent representations of comonotonicity. In particular, the third property gives a constructive method for building a comonotone probability measure $P_{X,Y}$ given marginals P_X and P_Y. For this aim, we just need to consider the marginal cdfs F_X and F_Y, then we define $F_{X,Y}(x, y) = \min\{F_X(x), F_Y(y)\}$ for every $(x, y) \in \mathscr{X} \times \mathscr{Y}$, and its associated probabiltiy measure $P_{X,Y}$ is comonotone. This ensures the existence of a joint comonotone probability measure with given marginals; in addition, such joint comonotone model is unique.

3.2. Comonotone Lower Probabilities

In a previous paper [13], the notion of comonotonicity was given for lower probabilities.

Definition 3 ([13]) *A coherent lower probability* $\underline{P}_{X,Y} : \mathscr{P}(\mathscr{X} \times \mathscr{Y}) \to [0,1]$ *is* comonotone *if every* $P \in \mathscr{M}(\underline{P}_{X,Y})$ *is comonotone.*

The idea behind this definition is that, from an epistemic point of view, $\mathscr{M}(\underline{P}_{X,Y})$ contains all the probability measures candidates for being a real but unknown probability P_0. If we know that P_0 is comonotone, it seems reasonable to require that all the probabilities in $\mathscr{M}(\underline{F}_{X,Y})$ are comonotone too.

In [13], it was also studied to which extent the properties in Theorem 2 are equivalent to the comonotonicity of lower probabilities.

Theorem 4 ([13]) *Let* $\underline{P}_{X,Y} : \mathscr{P}(\mathscr{X} \times \mathscr{Y}) \to [0,1]$ *be a coherent lower probability with conjugate* $\overline{P}_{X,Y}$*. The following statements are equivalent:*

1. $\underline{P}_{X,Y}$ *is a comonotone lower probability.*
2. *Supp*$(\underline{P}_{X,Y}) = \left\{(x,y) \in \mathscr{X} \times \mathscr{Y} \mid \overline{P}_{X,Y}(\{(x,y)\}) > 0\right\}$ *is an increasing set in* $\mathscr{X} \times \mathscr{Y}$.
3. *For every* $(x,y) \in \mathscr{X} \times \mathscr{Y}$, *either* $\overline{P}_{X,Y}\left(D_1^{(x,y)}\right) = 0$ *or* $\overline{P}_{X,Y}\left(D_2^{(x,y)}\right) = 0$.

Also, when the previous equivalent conditions hold, the bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ *associated with* $\underline{P}_{X,Y}$ *using Equations (2) and (3) can be expressed as:*

$$\begin{aligned} \underline{F}_{X,Y}(x,y) &= \min\left\{\underline{F}_X(x), \underline{F}_Y(y)\right\}, \\ \overline{F}_{X,Y}(x,y) &= \min\left\{\overline{F}_X(x), \overline{F}_Y(y)\right\} \quad \forall (x,y) \in \mathscr{X} \times \mathscr{Y}. \end{aligned} \tag{6}$$

However, the converse does not hold.

Comparing Theorems 2 and 4, we can see only one but very important difference: when the coherent lower probability is comonotone, its associated bivariate p-box can be expressed in terms of the marginals using the minimum operator, but the converse does not hold in general (see [13, Ex.19] for a counterexample).

This is the starting point of the research presented in this paper. The third item in Theorem 2 guarantees the existence and uniqueness of a comonotone probability measure with given marginals, and it also gives a constructive procedure. In this paper, we consider two marginal models given in terms of p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$, and we analyse whether:

(i) There is always a joint comonotone lower probability whose marginals are $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$. In case there is not, is it possible to determine which additional conditions must be imposed to guarantee its existence?

(ii) Can we give a constructive approach for building such a comonotone model (if it exists)?

(iii) In case it exists, is this joint comonotone model unique?

In the next section, we dig into these questions related to the existence, construction and uniqueness of a joint comonotone model with given marginal p-boxes.

4. Comonotone Extension

Consider two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ modelling our uncertainty about the marginals X and Y. We introduce the following terminology.

Definition 5 *Consider two marginal p-boxes* $(\underline{F}_X, \overline{F}_X)$ *and* $(\underline{F}_Y, \overline{F}_Y)$. *A coherent lower probability* $\underline{P}_{X,Y} : \mathscr{P}(\mathscr{X} \times \mathscr{Y}) \to [0,1]$ *is a* comonotone extension *of* $(\underline{F}_X, \overline{F}_X)$ *and* $(\underline{F}_Y, \overline{F}_Y)$ *if it is comonotone and its marginal p-boxes obtained using Equations (4) and (5) are* $(\underline{F}_X, \overline{F}_X)$ *and* $(\underline{F}_Y, \overline{F}_Y)$.

In the rest of this section we answer the questions related to the existence, construction and uniqueness of a comonotone extension $\underline{P}_{X,Y}$ of given marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$.

For the sake of simplicity, in what remains we consider the following notation and terminology:

- $(x_i, y_j) \prec (x_k, y_l)$ if $x_i \leq x_k$, $y_j \leq y_l$ and at least one of the inequalities is strict.
- The interval $[\underline{a}, \overline{a}]$ will be denoted, for short, as $\underline{\overline{a}}$.
- $\underline{\overline{a}}$ interval dominates $\underline{\overline{b}}$, denoted as $\underline{\overline{b}} \preceq \underline{\overline{a}}$, if $\underline{b} \leq \underline{a}$ and $\overline{b} \leq \overline{a}$. When at least one of the inequalities is strict, we speak about strict interval dominance, and denote it $\underline{\overline{b}} \prec \underline{\overline{a}}$.

4.1. Existence of a Comonotone Extension

We start answering the first question related to the existence of a comonotone extension with given marginal p-boxes. Unfortunately, the answer is negative.

Example 1 *Consider the possibility spaces* $\mathscr{X} = \{x_1, x_2\}$ *and* $\mathscr{Y} = \{y_1, y_2\}$ *and the marginal p-boxes* $(\underline{F}_X, \overline{F}_X)$ *and* $(\underline{F}_Y, \overline{F}_Y)$ *given by:*

$\mathscr{X}$	x_1	x_2
$\underline{F}_X(x_i)$	0	1
$\overline{F}_X(x_i)$	1	1

$\mathscr{Y}$	y_1	y_2
$\underline{F}_Y(y_i)$	0.5	1
$\overline{F}_Y(y_i)$	0.5	1

Note that $(\underline{F}_Y, \overline{F}_Y)$ *satisfies* $\underline{F}_Y = \overline{F}_Y$, *hence its credal set only contains one single probability* P_Y *given by* $P_Y(\{y_1\}) = P_Y(\{y_2\}) = 0.5$. *On the other hand, the credal* *set of associated with $(\underline{F}_X, \overline{F}_X)$ is formed by all the probability measures in $\mathbb{P}(\mathscr{X})$.*

If there exists a comonotone extension $\underline{P}_{X,Y}$ of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$, from Theorem 4 its associated bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ must satisfy Equation (6). Hence, $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ would be given by:

$\underline{\overline{F}}_X(x_i)$	$[0,1]$	$[1,1]$	
y_2	$[0,1]$	$[1,1]$	$[1,1]$
y_1	$[0,0.5]$	$[0.5,0.5]$	$[0.5,0.5]$
$\underline{\overline{F}}_{X,Y}(x_i,y_j)$	x_1	x_2	$\underline{\overline{F}}_Y(y_j)$

However, since $\overline{P}_{X,Y}$ is sub-additive [25, Sec.2.7.4.d], $\overline{P}_{X,Y}(A \cup B) \leq \overline{P}_{X,Y}(A) + \overline{P}_{X,Y}(B)$ for every $A, B \subseteq \mathscr{X} \times \mathscr{Y}$. Taking $A = \{(x_1,y_2)\}$ and $B = \{(x_1,y_1)\}$:

$$\begin{aligned}
&\overline{P}_{X,Y}\left(\{(x_1,y_2)\}\right) \\
&\quad \geq \overline{P}_{X,Y}\left(\{(x_1,y_1),(x_1,y_2)\}\right) - \overline{P}_{X,Y}\left(\{(x_1,y_1)\}\right) \\
&\quad = \overline{F}_{X,Y}(x_1,y_2) - \overline{F}_{X,Y}(x_1,y_1) = 1 - 0.5 = 0.5 > 0.
\end{aligned}$$

Also, from [25, Sec.2.7.4.d], $\underline{P}_{X,Y}(A \cup B) \leq \underline{P}_{X,Y}(A) + \overline{P}_{X,Y}(B)$ for every $A, B \subseteq \mathscr{X} \times \mathscr{Y}$. Taking $A = \{(x_1,y_1)\}$ and $B = \{(x_2,y_1)\}$, we obtain:

$$\begin{aligned}
&\overline{P}_{X,Y}\left(\{(x_2,y_1)\}\right) \\
&\quad \geq \underline{P}_{X,Y}\left(\{(x_1,y_1),(x_2,y_1)\}\right) - \underline{P}_{X,Y}\left(\{(x_1,y_1)\}\right) \\
&\quad = \underline{F}_{X,Y}(x_2,y_1) - \underline{F}_{X,Y}(x_1,y_1) = 0.5 - 0 = 0.5 > 0.
\end{aligned}$$

Hence, both (x_2,y_1) and (x_1,y_2) belong to $Supp(\underline{P}_{X,Y})$. Since these two elements are not comonotone, neither is $Supp(\underline{P}_{X,Y})$. According to Theorem 4, we conclude that $\underline{P}_{X,Y}$ is not comonotone. ♦

We conclude that not all the marginal p-boxes have a comonotone extension. In spite of this negative answer, our next theorem characterises the conditions that the marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ must satisfy in order to guarantee the existence of a comonotone extension.

Theorem 6 *Consider two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$. They have a comonotone extension $\underline{P}_{X,Y}$ if and only if for every $(x,y) \in \mathscr{X} \times \mathscr{Y}$ there is an interval dominance relation between $\underline{\overline{F}}_X(x)$ and $\underline{\overline{F}}_Y(y)$.*

This theorem characterises the conditions that the marginal p-boxes must satisfy in order for the comonotone extension to exist. In fact, the sufficient and necessary condition given in this theorem is quite simple: it only requires that $\underline{\overline{F}}_X(x) \preceq \underline{\overline{F}}_Y(y)$ or $\underline{\overline{F}}_Y(y) \preceq \underline{\overline{F}}_X(x)$ for every $(x,y) \in \mathscr{X} \times \mathscr{Y}$. As a matter of fact, we can see that the p-boxes in Example 1 do not satisfy this necessary and sufficient condition, because $\underline{F}_Y(y_1) < \underline{F}_X(x_1) = \overline{F}_X(x_1) < \overline{F}_Y(y_1)$, and that is why those p-boxes do not have a comonotone extension.

In the following subsection we give a constructive approach for building a comonotone extension.

4.2. Construction of a Comonotone Extension

Consider now two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ satisfying the sufficient and necessary conditions in Theorem 6, and let us determine a constructive method for building a comonotone extension. First of all, by Theorem 4, we know that if there exists a comonotone extension, its associated bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ is given by Equation (6). Considering that bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$, we define the following set[1]:

$$S = \Big\{ (x_i,y_j) \in \mathscr{X} \times \mathscr{Y} \mid \underline{\overline{F}}_{X,Y}(x_{i-1},y_j) \prec \underline{\overline{F}}_{X,Y}(x_i,y_j) \text{ and } \underline{\overline{F}}_{X,Y}(x_i,y_{j-1}) \prec \underline{\overline{F}}_{X,Y}(x_i,y_j) \Big\}. \quad (7)$$

This set satisfies some interesting properties, as we enumerate in the following lemma.

Lemma 7 *Consider two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$, and the bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ they define using Equation (6), as well as the set S defined in Equation (7). If for every $(x,y) \in \mathscr{X} \times \mathscr{Y}$ there is an interval dominance relation between $\underline{\overline{F}}_X(x) = \left[\underline{F}_X(x), \overline{F}_X(x)\right]$ and $\underline{\overline{F}}_Y(y) = \left[\underline{F}_Y(y), \overline{F}_Y(y)\right]$, then S satisfies the following properties:*

1. *S is an increasing set in $\mathscr{X} \times \mathscr{Y}$.*
2. *If $(x_i,y_j),(x_k,y_l) \in S$, $(x_i,y_j) \prec (x_k,y_l)$ and they are consecutive elements[2] in S, then any $(x,y) \in \mathscr{X} \times \mathscr{Y} \setminus S$ such that $(x_i,y_j) \prec (x,y) \prec (x_k,y_l)$ satisfies:*

$$\underline{F}_{X,Y}(x,y) = \underline{F}_{X,Y}(x_i,y_j), \; \overline{F}_{X,Y}(x,y) = \overline{F}_{X,Y}(x_i,y_j).$$

From this result we deduce that the set S is increasing and the bivariate p-box only increases in S. Consider now an increasing superset $S^* \supseteq S$ (as for example $S^* = S$). Since it is increasing, it can be rewritten as:

$$S^* = \{(u_1,v_1),\ldots,(u_s,v_s)\}, \quad (8)$$

where $(u_1,v_1) \prec \ldots \prec (u_s,v_s)$. This means that $(S,\prec)$ is a totally ordered space. Then, we can define a univariate possibility space $\mathscr{Z} = \{z_1,\ldots,z_s\}$ such that $z_1 < \ldots < z_s$. It is possible to establish a one-to-one correspondence between S^* and $\mathscr{Z}$ that identifies the elements (u_i,v_i) and z_i, for every $i = 1,\ldots,s$. Formally:

$$z_i = g(u_i,v_i), \text{ and } (u_i,v_i) = g^{-1}(z_i) \quad \forall i = 1,\ldots,n. \quad (9)$$

Now, we define a (univariate) p-box $(\underline{F}_Z, \overline{F}_Z)$ in $\mathscr{Z}$ as:

$$\underline{F}_Z(z_i) = \underline{F}_{X,Y}(u_i,v_i), \quad \overline{F}_Z(z_i) = \overline{F}_{X,Y}(u_i,v_i) \quad (10)$$

1. When $i = 1$ in Eq.(7), we assume that $\underline{F}_{X,Y}(x_{i-1},y_j) = \overline{F}_{X,Y}(x_{i-1},y_j) = 0$. Similarly, for $j = 1$ we assume $\underline{F}_{X,Y}(x_i,y_{j-1}) = \overline{F}_{X,Y}(x_i,y_{j-1}) = 0$.
2. This means that there is not other $(x^*,y^*) \in S$ such that $(x_i,y_j) \prec (x^*,y^*) \prec (x_k,y_l)$.

for every $i = 1, \dots, s$. The lower probability obtained as the lower envelope of the credal set $\mathscr{M}(\underline{F}_Z, \overline{F}_Z)$ is a belief function whose focal events are ordered intervals. The following result shows an interesting property of $\mathscr{M}(\underline{F}_Z, \overline{F}_Z)$.

Proposition 8 *Consider the marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ satisfying the condition in Lemma 7 and the bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ they define through Equation (6). Consider an increasing set S^* that is a superset of the set S defined in Equation (7), the correspondence in Equation (9) and the p-box $(\underline{F}_Z, \overline{F}_Z)$ defined in Equation (10). Then, there is a one-to-one correspondence between the credal sets:*

$$\mathscr{M}(\underline{F}_Z, \overline{F}_Z) = \{P_Z \in \mathbb{P}(\mathscr{Z}) \mid \underline{F}_Z \leq F_{P_Z} \leq \overline{F}_Z\} \text{ and}$$
$$\mathscr{M} = \{P_{X,Y} \in \mathbb{P}(\mathscr{X} \times \mathscr{Y}) \mid P_{X,Y}(S^*) = 1, \underline{F}_{X,Y} \leq F_{P_{X,Y}} \leq \overline{F}_{X,Y}\}.$$

The correspondence between both credal sets follows from the proof of this result, which has been omitted due to space limitations. On the one hand, for any $P_Z \in \mathscr{M}(\underline{F}_Z, \overline{F}_Z)$, we define $P_{X,Y} \in \mathscr{M}$ as:

$$P_{X,Y}(A) = P_Z\big(g(A \cap S^*)\big) \quad \forall A \subseteq \mathscr{X} \times \mathscr{Y}.$$

On the other hand, given $P_{X,Y} \in \mathscr{M}$, we define $P_Z \in \mathscr{M}(\underline{F}_Z, \overline{F}_Z)$ as:

$$P_Z(C) = P_{X,Y}\big(g^{-1}(C)\big) \quad \forall C \subseteq \mathscr{Z}.$$

Using this correspondence, if we denote by $\underline{P}_Z$ the belief function associated with $\mathscr{M}(\underline{F}_Z, \overline{F}_Z)$, we define a coherent lower probability on $\mathscr{P}(\mathscr{X} \times \mathscr{Y})$ by:

$$\underline{P}_{X,Y}(A) = \underline{P}_Z\big(g(A \cap S^*)\big) \quad \forall A \subseteq \mathscr{X} \times \mathscr{Y}. \tag{11}$$

Theorem 9 *In the conditions of Proposition 8, the lower probability $\underline{P}_{X,Y}$ in Equation (11) is a comonotone extension of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$, and it is a belief function. Moreover, the basic probability assignment of $\underline{P}_{X,Y}$ is given by:*

$$m(A) = \begin{cases} 0 & \text{if } A \not\subseteq S^*, \\ m_Z\big(g(A)\big) & \text{if } A \subseteq S^*, \end{cases} \tag{12}$$

for every $A \subseteq \mathscr{X} \times \mathscr{Y}$.

The steps described above lead us to a constructive procedure for building a comonotone extension of the marginal p-boxes.

Example 2 *Consider the possibility spaces $\mathscr{X} = \{x_1, x_2, x_3\}$ and $\mathscr{Y} = \{y_1, y_2, y_3\}$, and the p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ given by:*

$\mathscr{X}$	x_1	x_2	x_3
$\underline{F}_X(x_i)$	0	0.4	1
$\overline{F}_X(x_i)$	0.2	0.8	1

$\mathscr{Y}$	y_1	y_2	y_3
$\underline{F}_Y(y_j)$	0.1	0.4	1
$\overline{F}_Y(y_j)$	0.4	0.8	1

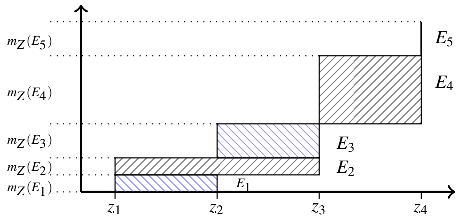

Figure 1: P-box $(\underline{F}_Z, \overline{F}_Z)$ in Example 2.

They satisfy the necessary and sufficient condition in Theorem 6 because, for every $i = 1,2,3$ and $j = 1,2,3$, there is an interval dominance relation between $\underline{\overline{F}}_X(x_i)$ and $\underline{\overline{F}}_Y$:

$$\underline{\overline{F}}_X(x_1) = [0, 0.2] \prec \underline{\overline{F}}_Y(y_1) = [0.1, 0.4] \prec$$
$$\underline{\overline{F}}_X(x_2) = \underline{\overline{F}}_Y(y_2) = [0.4, 0.8] \prec \underline{\overline{F}}_X(x_3) = \underline{\overline{F}}_Y(y_3) = [1, 1].$$

The bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ they define through Equation (6) is given by:

$\underline{\overline{F}}_X(x_i)$	$[0, 0.2]$	$[0.4, 0.8]$	$[1, 1]$	
y_3	$[0, 0.2]$	$[0.4, 0.8]$	$[1, 1]$	$[1, 1]$
y_2	$[0, 0.2]$	$[0.4, 0.8]$	$[0.4, 0.8]$	$[0.4, 0.8]$
y_1	$[0, 0.2]$	$[0.1, 0.4]$	$[0.1, 0.4]$	$[0.1, 0.4]$
$\underline{\overline{F}}_{X,Y}(x_i, y_j)$	x_1	x_2	x_3	$\underline{\overline{F}}_Y(y_j)$

Let us compute the set S given in Equation (7). First of all, since $\overline{F}_{X,Y}(x_1, y_1) = 0.2 > 0$, $(x_1, y_1) \in S$. Next:

$$\underline{\overline{F}}_{X,Y}(x_1, y_1) = [0, 0.2] \prec \underline{\overline{F}}_{X,Y}(x_2, y_1) = [0.1, 0.4],$$

whence $(x_2, y_1) \in S$. With a similar reasoning, we can see that (x_2, y_2) and (x_3, y_3) also belong to S. This means that the set S, highlighted in blue in the previous table, is given by:

$$S = \{(x_1, y_1), (x_2, y_1), (x_2, y_2), (x_3, y_3)\}.$$

From Lemma 7, this set is increasing, so it can be rewritten as:

$$S = \{(u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4)\},$$

where $(u_i, v_i) \prec (u_{i+1}, v_{i+1})$ for $i = 1,2,3$. Considering $S^* = S$, we define the possibility space $\mathscr{Z} = \{z_1, z_2, z_3, z_4\}$ with $z_1 < z_2 < z_3 < z_4$, and the correspondence in Equation (9). Using also Equation (10), we define the p-box $(\underline{F}_Z, \overline{F}_Z)$:

$\mathscr{Z}$	z_1	z_2	z_3	z_4
$\underline{F}_Z(z_i)$	0	0.1	0.4	1
$\overline{F}_Z(z_i)$	0.2	0.4	0.8	1

This p-box, as well as the focal events of its induced belief function, have been graphically depicted in Figure 1. These focal events are given by:

	E_1	E_2	E_3	E_4	E_5
E_i	$\{z_1, z_2\}$	$\{z_1, z_2, z_3\}$	$\{z_2, z_3\}$	$\{z_3, z_4\}$	$\{z_4\}$
$m_Z(E_i)$	0.1	0.1	0.2	0.4	0.2

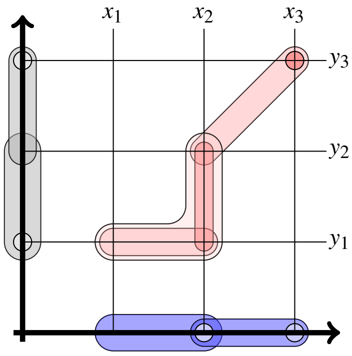

Figure 2: Graphical representation of the focal events of the belief function $\underline{P}_{X,Y}$ in Example 2.

Using Equation (12), the focal events of $\underline{P}_{X,Y}$ *are given by:*

	F_1	F_2	F_3	F_4	F_5
$F_i = g^{-1}(E_i)$	(x_1,y_1) (x_2,y_1)	(x_1,y_1) (x_2,y_1) (x_2,y_2)	(x_2,y_1) (x_2,y_2)	(x_2,y_2) (x_3,y_3)	(x_3,y_3)
$m(F_i) = m_Z(E_i)$	0.1	0.1	0.2	0.4	0.2

These focal events, together with the focal events of the marginal p-boxes, are graphically depicted in Figure 2. Of course, the support of the belief function $\underline{P}_{X,Y}$ *coincides with the given set* $S^* = S$*, which is increasing. Since the marginal p-boxes of* $\underline{P}_{X,Y}$ *are* $(\underline{F}_X,\overline{F}_X)$ *and* $(\underline{F}_Y,\overline{F}_Y)$*, we conclude that* $\underline{P}_{X,Y}$ *is a comonotone extension of the marginal p-boxes.* ♦

4.3. Uniqueness of the Comonotone Extension

We have already solved the problem of the existence and construction of a comonotone extension with given marginal p-boxes. However, we still need to study the uniqueness of such comonotone extension. As we show in our next example, that is a follow-up of our on-going example, that the comonotone extension is not unique in general.

Example 3 *Let us continue with Example 2. There, we have built a comonotone extension* $\underline{P}_{X,Y}$ *of the marginal p-boxes* $(\underline{F}_X,\overline{F}_X)$ *and* $(\underline{F}_Y,\overline{F}_Y)$ *using the increasing set* $S^* = S$*.*

However, instead of considering the increasing set S, as we did in Example 2, we can consider the increasing supersets

$$S_1^* = \{(x_1,y_1),(x_2,y_1),(x_2,y_2),(x_2,y_3),(x_3,y_3)\},$$
$$S_2^* = \{(x_1,y_1),(x_2,y_1),(x_2,y_2),(x_3,y_2),(x_3,y_3)\}.$$

Following the procedure described in Section 4.2 with S_1^* *and* S_2^**, we obtain two other belief functions,* $\underline{P}_1$ *and* $\underline{P}_2$*, which are also comonotone extensions of* $(\underline{F}_X,\overline{F}_X)$ *and* $(\underline{F}_Y,\overline{F}_Y)$*. Their basic probability assignments,* m_1 *and* m_2*, and focal events are given by:*

	G_1	G_2	G_3	G_4	G_5
G_i	(x_1,y_1) (x_2,y_1)	(x_1,y_1) (x_2,y_1) (x_2,y_2)	(x_2,y_1) (x_2,y_2)	(x_2,y_2) (x_3,y_3) (x_2,y_3)	(x_3,y_3)
$m_1(G_i)$	0.1	0.1	0.2	0.4	0.2

	H_1	H_2	H_3	H_4	H_5
H_i	(x_1,y_1) (x_2,y_1)	(x_1,y_1) (x_2,y_1) (x_2,y_2)	(x_2,y_1) (x_2,y_2)	(x_2,y_2) (x_3,y_3) (x_3,y_2)	(x_3,y_3)
$m_2(H_i)$	0.1	0.1	0.2	0.4	0.2

These focal events are depicted in Figure 3. Both belief functions are also comonotone extensions of $(\underline{F}_X,\overline{F}_X)$ *and* $(\underline{F}_Y,\overline{F}_Y)$*. Hence, we have three different comonotone extensions of the marginals p-boxes,* $\underline{P}_{X,Y}$*,* $\underline{P}_1$ *and* $\underline{P}_2$*, so the comonotone extension is not unique.* ♦

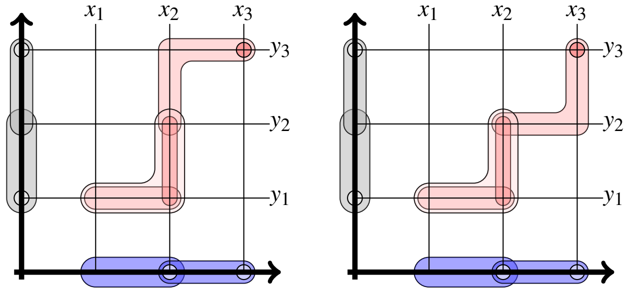

Figure 3: Graphical representation of the focal events of the comonotone extensions $\underline{P}_1$ (left picture) and $\underline{P}_2$ (right picture) of $(\underline{F}_X,\overline{F}_X)$ and $(\underline{F}_Y,\overline{F}_Y)$ in Example 3.

This example shows that we cannot guarantee the uniqueness of a comonotone extension of the given marginal p-boxes. It is then of interest to study the existence of a least-committal comonotone extension of the p-boxes. This will be done in Section 5.

5. Comonotone Natural Extension

Examples 2 and 3 show that the comonotone extension of two marginal p-boxes is not unique in general. When the comonotone extension exists but it is not unique, we can follow the usual procedure in the imprecise probability literature: the *natural extension* [25]. The natural extension is the least committal extension, using only the available information and adding no extra information. This is done for example in the cases of independence, with the independent natural extension [4], conglomerability, with the conglomerable natural extension [12], or the extension of marginal models with no information about their dependence [16, Sec.3.2].

Definition 10 *Consider two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$. A coherent lower probability $\underline{E}_{X,Y} : \mathscr{P}(\mathscr{X} \times \mathscr{Y}) \to [0,1]$ is the* comonotone natural extension *of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ if it is a comonotone extension of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ and it is the least-committal extension.*

Of course, if the comonotone natural extension exists, $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ satisfy the condition in Theorem 6. However, the next example shows that this necessary condition is not sufficient in general.

Example 4 *Consider the same setting as in Examples 2–3. Example 3 shows two comonotone extensions $\underline{P}_1$ and $\underline{P}_2$ of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$. Ex-absurdo, if there exists a comonotone natural extension $\underline{E}_{X,Y}$ of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$, then by definition $\overline{E}_{X,Y} \geq \overline{P}_1, \overline{P}_2$, where of course $\overline{E}_{X,Y}$, $\overline{P}_1$ and $\overline{P}_2$ denote the conjugate upper probabilities of $\underline{E}$, $\underline{P}_1$ and $\underline{P}_2$, respectively. From this inequality we deduce that:*

$$\overline{E}_{X,Y}\big(\{(x_2,y_3)\}\big) \geq \overline{P}_1\big(\{(x_2,y_3)\}\big) > 0,$$
$$\overline{E}_{X,Y}\big(\{(x_3,y_2)\}\big) \geq \overline{P}_2\big(\{(x_3,y_2)\}\big) > 0.$$

If follows that both (x_2,y_3) and (x_3,y_2) belong to $Supp(\underline{E}_{X,Y})$, so the support would not be increasing. This means that $\underline{E}_{X,Y}$ would not be comonotone. ♦

From this example we deduce that the comonotone natural extension of two marginal p-boxes does not always exist, even when there are comonotone extensions. Interestingly, the example above also shows that the comonotone extension is not preserved when taking lower envelopes of comonotone extensions, unlike what happens with the properties of independence or coherence, for instance.

Our aim now is to investigate which additional condition must be imposed to guarantee the existence of the comonotone natural extension and, in such case, how to build it.

Consider again the set S defined in Equation (7), and rewrite it as in Equation (8). For each $(u_l, v_l) \in S$, we define:

$$S_l = \begin{cases} \{(x,y) \in \mathscr{X} \times \mathscr{Y} \mid x = u_l,\ y \in [v_l, v_{l+1})\}, & \text{if } \overline{F}_Y(v_l) \prec \overline{F}_X(u_l), \\ \{(x,y) \in \mathscr{X} \times \mathscr{Y} \mid x \in [u_l, u_{l+1}),\ y = v_l\}, & \text{if } \overline{F}_X(u_l) \prec \overline{F}_Y(v_l), \\ \{(u_l, v_l)\}, & \text{if } \underline{F}_X(u_l) = \overline{F}_X(u_l) = \underline{F}_Y(v_l) = \overline{F}_Y(v_l), \end{cases}$$

for every $l = 1, \ldots, s$. These sets satisfy the following properties:

Lemma 11 *The sets $S_1, \ldots, S_s$ are disjoint and $S^* = \cup_{l=1}^s S_l$ is an increasing superset of S.*

From this technical result we deduce that the set $S^* = \cup_{l=1}^s S_l$ can be used for defining a comonotone extension following the steps described in Section 4.2. More importantly, using this increasing superset of S we can build the comonotone natural extension, as next theorem shows.

Theorem 12 *There exists a comonotone natural extension of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ if and only if for every $(x,y) \in \mathscr{X} \times \mathscr{Y}$, either there is a strict interval dominance relationship between $\overline{F}_X(x)$ and $\overline{F}_Y(y)$ or $\underline{F}_X(x) = \overline{F}_X(x) = \underline{F}_Y(y) = \overline{F}_Y(y)$.*

Moreover, when this holds, the comonotone natural extension $\underline{E}_{X,Y}$ is the belief function built in Section 4.2 using the set $S^* = \cup_{l=1}^s S_l$.

This theorem not only characterises the conditions that the marginal p-boxes must satisfy to guarantee the existence of their comonotone natural extension, but also gives the constructive method for building it and also assures that the comonotone natural extension is a belief function. It is worth mentioning that the p-boxes in Example 2 do not satisfy the condition given in Theorem 12, since $\overline{F}_X(x_2) = \overline{F}_Y(y_2) = [0.4, 0.8]$, and for this reason we saw in Example 4 that the comonotone natural extension of these p-boxes does not exist.

Example 5 *Consider the possibility spaces $\mathscr{X} = \{x_1, x_2, x_3\}$ and $\mathscr{Y} = \{y_1, y_2, y_3, y_4\}$, and the p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ given by:*

$\mathscr{X}$	x_1	x_2	x_3
$\underline{F}_X(x_i)$	0	0.8	1
$\overline{F}_X(x_i)$	0.5	0.8	1

$\mathscr{Y}$	y_1	y_2	y_3	y_4
$\underline{F}_Y(y_j)$	0.2	0.2	0.8	1
$\overline{F}_Y(y_j)$	0.8	0.8	0.8	1

The condition in Theorem 12 is satisfied because:

$$\overline{F}_X(x_1) = [0, 0.5] \prec \overline{F}_Y(y_1) = \overline{F}_Y(y_2) = [0.2, 0.8] \prec$$
$$\overline{F}_X(x_2) = \overline{F}_Y(y_3) = [0.8, 0.8] \prec \overline{F}_X(x_4) = \overline{F}_Y(y_4) = [1,1].$$

The bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ they define through Equation (6) applying the minimum operator to the lower and upper bounds, respectively, is given by:

$\overline{F}_X(x_i)$	$[0, 0.5]$	$[0.8, 0.8]$	$[1,1]$	
y_4	$[0, 0.5]$	$[0.8, 0.8]$	$[1,1]$	$[1,1]$
y_3	$[0, 0.5]$	$[0.8, 0.8]$	$[0.8, 0.8]$	$[0.8, 0.8]$
y_2	$[0, 0.5]$	$[0.2, 0.8]$	$[0.2, 0.8]$	$[0.2, 0.8]$
y_1	$[0, 0.5]$	$[0.2, 0.8]$	$[0.2, 0.8]$	$[0.2, 0.8]$
$\overline{F}_{X,Y}(x_i, y_j)$	x_1	x_2	x_3	$\overline{F}_Y(y_j)$

The set S defined in Equation (7) is given by:

$$S = \{(x_1,y_1), (x_2,y_1), (x_2,y_3), (x_3,y_4)\}$$
$$= \{(u_1,v_1), \ldots, (u_4,v_4)\}.$$

Since $\overline{F}_X(u_1) \prec \overline{F}_Y(v_1)$, the set S_1 is given by:

$$S_1 = \big\{(x,y) \in \mathscr{X} \times \mathscr{Y} \mid x \in [u_1, u_2), y = v_1\big\} = \{(x_1, y_1)\}.$$

Similarly, the set S_2 is given by:

$$S_2 = \{(x,y) \in \mathscr{X} \times \mathscr{Y} \mid x = u_2, y \in [v_2, v_3)\}$$
$$= \{(x_2,y_1), (x_2,y_2)\}.$$

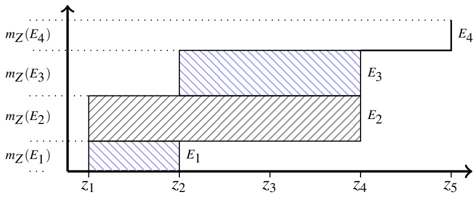

Figure 4: P-box $(\underline{F}_Z, \overline{F}_Z)$ defined in Equation (13) with the focal events of its associated belief function.

Iterating the procedure, we obtain $S_3 = \{(x_2, y_3)\}$ *and* $S_4 = \{(x_3, y_4)\}$*, hence* $S^* = \cup_{l=1}^{4} S_l$ *is given by:*

$$S^* = \{(x_1,y_1),(x_2,y_1),(x_2,y_2),(x_2,y_3),(x_3,y_4)\}.$$

Let us apply the procedure in Section 4.2 using S^**. For this aim, consider the possibility space* $\mathscr{Z} = \{z_1, z_2, z_3, z_4, z_5\}$ *and the univariate p-box defined in Equation (10):*

$\mathscr{Z}$	z_1	z_2	z_3	z_4	z_5
$\underline{F}_Z$	0	0.2	0.2	0.8	1
$\overline{F}_Z$	0.5	0.8	0.8	0.8	1

(13)

Its graphical representation is shown in Figure 4, where we also show the focal events of the belief function $\underline{P}_Z$ *associated with* $(\underline{F}_Z, \overline{F}_Z)$*, which are given by:*

	E_1	E_2	E_3	E_4
E_i	$\{z_1, z_2\}$	$\{z_1, z_2, z_3, z_4\}$	$\{z_2, z_3, z_4\}$	$\{z_5\}$
$m_Z(E_i)$	0.2	0.3	0.3	0.2

Using Equation (12), the focal events of $\underline{E}_{X,Y}$ *are given by:*

	F_1	F_2	F_3	F_4
$F_i = g^{-1}(E_i)$	(x_1,y_1) (x_2,y_1)	(x_1,y_1) (x_2,y_2) (x_2,y_1) (x_2,y_3)	(x_2,y_1) (x_2,y_3) (x_2,y_2)	(x_3,y_4)
$m(F_i) = m_Z(E_i)$	0.2	0.3	0.3	0.2

These focal events have been graphically depicted in Figure 5. According to our previous result, this is the comonotone natural extension of the given marginal p-boxes. ♦

6. Conclusions

The notion of comonotonicity for lower probabilities was introduced in [13], where their main properties were also studied. In this paper we have solved one of the open questions proposed in that paper, related to the construction of a joint comonotone model with given marginals. In this paper, we have considered that the marginals are given in terms of p-boxes, and we have analysed the problems of the existence, construction and uniqueness of a comonotone extension. Even if a first attempt could be to consider all the compatible comonotone models built using the marginals p-boxes and taking a lower envelope, Example 4 shows that this approach is not useful.

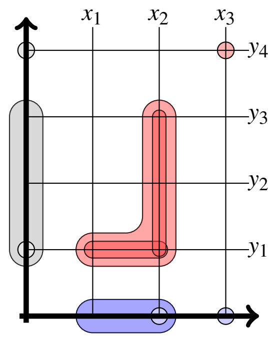

Figure 5: Graphical representation of the focal events of the comonotone natural extension $\underline{E}_{X,Y}$ in Example 2.

From our results, we deduce that (i) we cannot always guarantee the existence of a comonotone model, but we have characterised when such comonotone model exists. In fact, its existence depends on the interval dominance between the values of the marginal p-boxes; (ii) when it exists, we have seen a constructive method for building it; and (iii) when it exists, we have seen that such comonotone extension may not be unique. The lack of uniqueness led us to investigate the existence of a most conservative comonotone extension. We have seen that in general such most conservative comonotone extension does not always exists, but we have characterised its existence, giving rise to the notion of comonotone natural extension.

The dual dependence structure to comonotonicity is countermonotonicity, which refers to random variable with a negative dependence. It is well know that X and Y are comonotone if and only if X and $-Y$ are countermonotone, hence our results about comonotonicity can be straightforwardly extended to countemonotonicity.

There are still a number of interesting open problems related to comonotonicity for imprecise models. The former is the analysis of the existence, construction and uniqueness of a comonotone extension when the marginal models are given in terms of coherent lower probabilities instead of p-boxes. In addition, one of the fields of application of comonotonicity is finance [6, 7], so it would be interesting to analyse the extent to our results about comonotonicity with imprecise models can be used in this field.

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