Verified Interactive Computation of Definite Integrals

Runqing Xu, Liming Li, Bohua Zhan

SKLCS, Institute of Software, Chinese Academy of Sciences

Motivation

Many symbolic computations are involved in science and engineering, including reasoning about safety-critical systems.

If we let F be an exponential signal, the resulting response satisfies

$$M_t s^2\, p - mls^2\,\theta + cs\, p = F,$$

$$J_t s^2\,\theta - mls^2\, p + \gamma\, s\,\theta - mgl\,\theta = 0,$$

where all signals are exponential signals. The resulting transfer functions for the position of the cart and the orientation of the pendulum are given by solving for p and θ in terms of F to obtain

$$H_{\theta F} = \frac{mls}{(M_t J_t - m^2 l^2)s^4 + (\gamma\, M_t + cJ_t)s^2 + (c\gamma\, - M_t mgl)s - mglc},$$

$$H_{pF} = \frac{J_t s^2 + \gamma\, s - mgl}{(M_t J_t - m^2 l^2)s^4 + (\gamma\, M_t + cJ_t)s^3 + (c\gamma\, - M_t mgl)s^2 - mglcs},$$

where each of the coefficients is positive. The pole zero diagrams for these two transfer functions are shown in Figure 8.5 using the parameters from Example 6.7.

If we assume the damping is small and set $c = 0$ and $\gamma\, = 0$, we obtain

$$H_{\theta F} = \frac{ml}{(M_t J_t - m^2 l^2)s^2 - M_t mgl},$$

$$H_{pF} = \frac{J_t s^2 - mgl}{s^2\big((M_t J_t - m^2 l^2)s^2 - M_t mgl\big)}.$$

This gives nonzero poles and zeros at

$$p = \pm\sqrt{\frac{mglM_t}{M_t J_t - m^2 l^2}} \approx \pm 2.68, \qquad z = \pm\sqrt{\frac{mgl}{J_t}} \approx \pm 2.09.$$

2.457

1. $$\int \frac{(A + B\cosh x)\,dx}{\cosh x\,(a + b\cosh x)} = \frac{1}{a}\left[A \arctan\sinh x - (Ab - Ba)\int \frac{dx}{a + b\cosh x}\right]$$

(see **2.443** 3)

2.458

1. $$\int \frac{dx}{a + b\sinh^2 x}$$

$$= \frac{1}{\sqrt{a(b-a)}} \arctan\left(\sqrt{\frac{b}{a} - 1}\tanh x\right) \qquad \left[\frac{b}{a} > 1\right]$$

$$= \frac{1}{\sqrt{a(a-b)}} \operatorname{arctanh}\left(\sqrt{1 - \frac{b}{a}}\tanh x\right) \qquad \left[0 < \frac{b}{a} < 1 \text{ or } \frac{b}{a} < 0 \text{ and } \sinh^2 x < -\frac{a}{b}\right]$$

$$= \frac{1}{\sqrt{a(a-b)}} \operatorname{arccoth}\left(\sqrt{1 - \frac{b}{a}}\tanh x\right) \qquad \left[\frac{b}{a} < 0 \text{ and } \sinh^2 x > -\frac{a}{b}\right]$$

MZ 195

2. $$\int \frac{dx}{a + b\cosh^2 x}$$

$$= \frac{1}{\sqrt{-a(a+b)}} \arctan\left(\sqrt{-\left(1 + \frac{b}{a}\right)}\coth x\right) \qquad \left[\frac{b}{a} < -1\right]$$

$$= \frac{1}{\sqrt{a(a+b)}} \operatorname{arctanh}\left(\sqrt{1 + \frac{b}{a}}\coth x\right) \qquad \left[-1 < \frac{b}{a} < 0 \text{ and } \cosh^2 x > -\frac{a}{b}\right]$$

$$= \frac{1}{\sqrt{a(a+b)}} \operatorname{arccoth}\left(\sqrt{1 + \frac{b}{a}}\coth x\right) \qquad \left[\frac{b}{a} > 0 \text{ or } -1 < \frac{b}{a} < 0 \text{ and } \cosh^2 x < -\frac{a}{b}\right]$$

MZ 202

Motivation

In this work, we consider definite integrals on a finite interval.

Q: find the error in the following calculation:

$$\int_0^{\pi} \sqrt{1+\cos(2x)}\,dx = \int_0^{\pi} \sqrt{1+\cos^2(x)-\sin^2(x)}\;dx$$
$$= \int_0^{\pi} \sqrt{2\cos^2(x)}\;dx$$
$$= \int_0^{\pi} \sqrt{2}\cos(x)\,dx = \sqrt{2}\sin(\pi) - \sqrt{2}\sin(0) = 0$$

A: $\sqrt{2\cos^2(x)}$ should be simplified to $\sqrt{2}|\cos(x)|$. The correct answer is $2\sqrt{2}$.

On Python's SymPy 1.5.1:

```
In [1]: from sympy import *
        from sympy.abc import x
        integrate(sqrt(1+cos(2*x)), (x, 0, pi))

Out[1]: 0
```

Motivation

How to guarantee the correctness of symbolic computation?

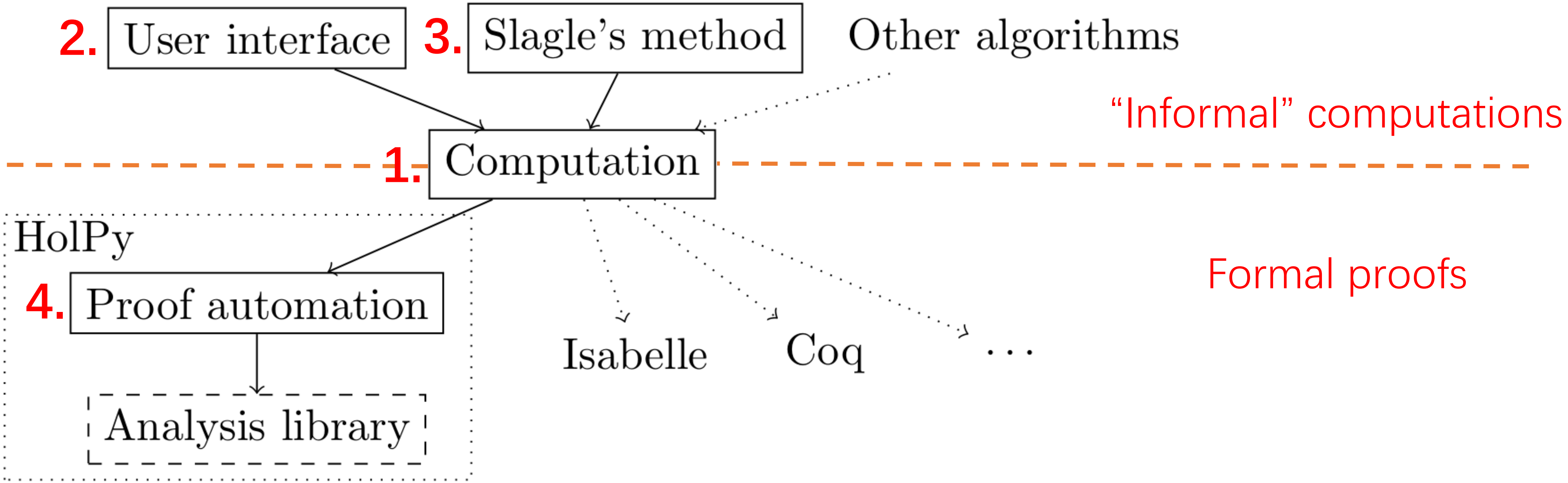
1. Use computer algebra systems, such as Mathematica and Maple:
   - Advantage: easy to use and automatic.
   - Disadvantage: cannot solve everything, **cannot actually guarantee correctness** (Durán et al, 2014)
2. Use interactive theorem provers, such as Isabelle/HOL, Coq, HOL Light, HOL4...
   - Advantage: strong guarantees of correctness.
   - Disadvantage: not automatic, requires users to be proficient in higher-order logic, analysis library, etc.

Can we combine the advantages of computer algebra systems and ITP?

Architecture

Goal: let user perform computations in a familiar CAS-like setting, then convert the computation into higher-order logic proofs.

Overall plan: fix an intermediate language of computational rules, as a bridge between formal and "informal".

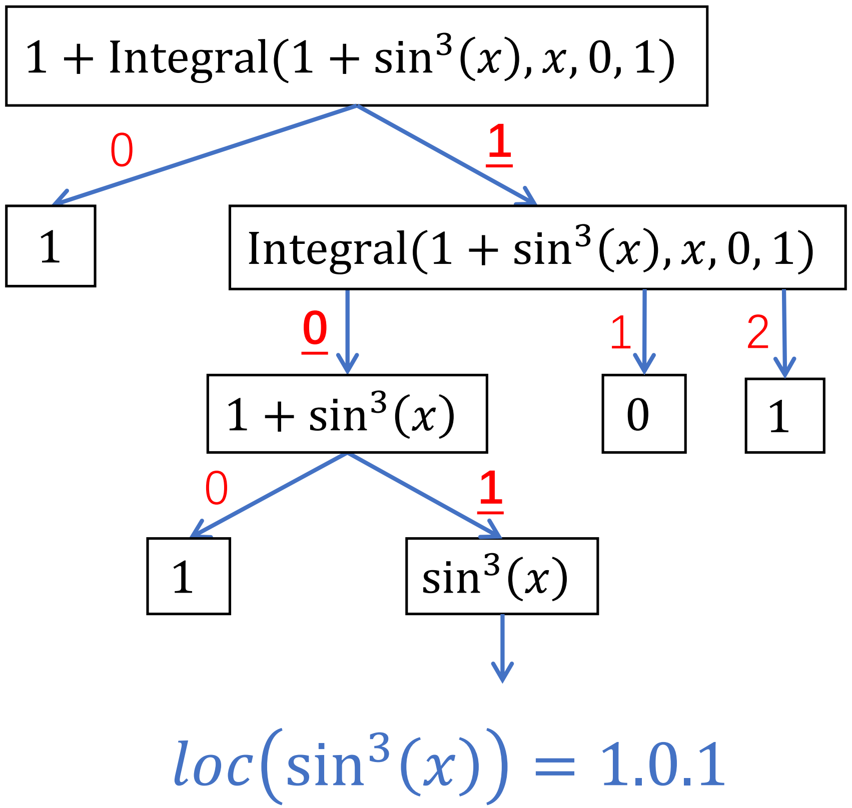


Terms and Computations

- Syntax:

$$e \ ::= v \mid c \mid e_1 \; op \; e_2 \mid f(e) \mid \mathrm{Deriv}(e) \mid \mathrm{Integral}(e, a, b)$$

- Location: point to subexpressions by specifying the path to a subtree in the AST of the expression.



- A computation is represented as a list of steps, each step containing:
  - which rule is used;
  - location the rule is applied;
  - parameters for the rule;
  - result of computation.
- Around 10 kinds of steps in current work.

Integration Rules

Simplification: rewrite an expression to an equivalent simpler form.

- Combine terms:

$$\pi + \frac{\pi}{3} \Longrightarrow \frac{4\pi}{3}$$

- Expand products of polynomials:

$$(x+1)\times(x-1) \Longrightarrow x^2 - 1$$

- Simplify values of trigonometric functions:

$$\sin\left(\frac{\pi}{6}\right) \Longrightarrow \frac{1}{2} \qquad \sin\left(\frac{\pi}{2} - x\right) \Longrightarrow \cos(x)$$

- Basic integrals:

$$\int_0^1 6x\,dx \Longrightarrow 3 \qquad \int_1^2 \frac{2}{x}\,dx \Longrightarrow 2\log(2) \qquad \int_0^{\frac{\pi}{2}} \cos(x)\,dx \Longrightarrow 1$$

Not included:

- **Expansion of large powers (e.g. $(x+y+1)^7$).**
- **Reduce quotients of polynomials (e.g. $\frac{x^2-1}{x-1} = x+1$).**

Integration Rules

Substitution: apply $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$ in either direction.

Forward substitution:

$$\int_a^b f(g(x))g'(x)dx \Rightarrow \int_{g(a)}^{g(b)} f(u)du$$

Example:

$$\int_{3/4}^{1} \frac{1}{\sqrt{1-x}-1}\,dx$$

$$= \int_0^{1/2} \frac{2u}{u-1}du \quad (\text{substitute } u \text{ for } \sqrt{1-x})$$

Record both f and g as parameters.

Here $f = \frac{2u}{u-1}$ and $g = \sqrt{1-x}$.

Backward Substitution:

$$\int_{g(a)}^{g(b)} f(u)du \Rightarrow \int_a^b f(g(x))g'(x)dx$$

Example:

$$\int_0^1 \sqrt{1-x^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2(t)}\cos(t)\,dt \quad (\text{substitute } x \text{ by } \sin(t))$$

Record g, a, b as parameters.

Here $g = \sin(t)$, $a = 0$, $b = \pi/2$.

Integration Rules

Trigonometric Identities: rewrite an expression to a possibly more complex form, in order to prepare for a substitution or integration by parts.

Fu et al. divides the trigonometric identities into several groups, with name of the form **TR**i, for example:

- TR5: $\sin^2(x) = 1 - \cos^2(x)$
- TR7: $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- TR9: $\sin(x) + \cos(x) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$, etc
- TR11: $\sin(2x) = 2\sin(x)\cos(x)$, $\cos(2x) = \cos^2(x) - \sin^2(x)$, etc

In the computation step, record name of Fu's rule and location of application.

Example:

$$\int_{\pi/6}^{\pi/2} \cos^2(x)\, dx = \int_{\pi/6}^{\pi/2} \frac{\cos(2x)+1}{2}\, dx \quad \text{(Trigonometric identity, TR7)}$$

Integration Rules

Integration by Parts:

$$\int_a^b u(x)v'(x)dx = u(x)v(x)|_a^b - \int_a^b u'(x)v(x)dx$$

Example:

$\int_{-1}^{2} xe^x dx = xe^x|_{-1}^{2} - \int_{-1}^{2} e^x dx$ (Integration by parts, $u = x$, $v = e^x$)

Splitting an integral:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad (a \le c \le b)$$

Example:

$\int_{-1}^{1} \sqrt{x^2}\, dx = \int_{-1}^{0} \sqrt{x^2}\, dx + \int_{0}^{1} \sqrt{x^2}\, dx$ (Splitting an integral, $c = 0$)

User Interface

Allows user to specify computation steps, provides various conveniences:

- Display in LaTeX format.
- Selection of actions and subexpressions to perform the action on.
- Automatically generate some parameters of steps (e.g. name of Fu's rule).
- Automatic integration algorithms (Slagle's method).
- Conversion to higher-order logic, showing ✓ if successful.

Integral File ▾ Calc ▾ Actions ▾

Exercise 1:

$\int_0^{\pi/100} \frac{\sin(20x)+\sin(19x)}{\cos(20x)+\cos(19x)}\,dx$

Exercise 2:

$\int_0^{1} \frac{e^x+\cos(x)}{e^x+\sin(x)}\,dx$

Step 1: $\int_0^{\pi/100} \frac{\sin(20x)+\sin(19x)}{\cos(20x)+\cos(19x)}\,dx$ Initial

Step 2: $\int_0^{\pi/100} \frac{\sin(\frac{1}{2}\times 39x)}{\cos(\frac{1}{2}\times 39x)}\,dx$ Rewrite trigonometric $\frac{\sin(20x)+\sin(19x)}{\cos(20x)+\cos(19x)}$ to $\frac{\sin(\frac{1}{2}\times 39x)}{\cos(\frac{1}{2}\times 39x)}$ ✓

Step 3: $\int_0^{39/200\pi} \frac{\frac{2}{39}}{\cos(u)}\sin(u)\,du$ Substitute u for $\frac{1}{2}\times 39x$ ✓

Step 4: $\int_{\cos(39/200\pi)}^{1} \frac{\frac{2}{39}}{v}\,dv$ Substitute v for $\cos(u)$ ✓

Step 5: $-\frac{2}{39}\log(\cos(\frac{39}{200}\pi))$ Simplification ✓

User Interface: Example

Integral File Calc Actions

$\int_1^2 x^2 + \frac{1}{x^4}\, dx$

Exercise 8:
$\int_{\pi/3}^{\pi} \sin\left(2x + \frac{\pi}{3}\right) dx$

Exercise 9:
$\int_4^9 \sqrt[3]{x}\left(\sqrt{x} + 1\right) dx$

Exercise 10:
$\int_{-1}^{0} \frac{3x^4+3x^2+1}{x^2+1}\, dx$

Exercise 11:
$\int_4^{e+3} \frac{x^3-12x^2-42}{x-3}\, dx$

Exercise 12:
$\int_0^{\pi/2} \sin(x) \cos^3(x)\, dx$

Exercise 13:
$\int_0^{\pi} 1 - \sin^3(x)\, dx$

Exercise 14:
$\int_{\pi/6}^{\pi/2} \cos^2(x)\, dx$

Exercise 15:
$\int_0^1 \sqrt{1-x^2}\, dx$

Step 1: $\int_{\pi/6}^{\pi/2} \cos^2(x)\, dx$ Initial

1: $\int_{\pi/6}^{\pi/2} \cos^2(x)\, dx$

Close

1: $\int_{\pi/6}^{\pi/2} \cos^2(x)\, dx$

2: $\int_{\pi/6}^{\pi/2} \frac{\cos(2x)}{2} + \frac{1}{2}\, dx$

3: $\int_{\pi/6}^{\pi/2} 1 - \sin^2(x)\, dx$

Slagle's Method

A heuristic integration algorithm (Slagle, 1963)

- Simple but effective.
- Output is human readable, can be translated to computation steps.
- Search on algorithmic and heuristic transformations, maintaining a tree consisting of AND nodes and OR nodes.

AND node

(all child nodes must be resolved)

$$\int_0^1 x^3 + x^2 + x\, dx$$

$$\int_0^1 x^3\, dx \qquad \int_0^1 x^2\, dx \qquad \int_0^1 x\, dx$$

OR node

(one child node need resolved)

$$\int_0^{\frac{\pi}{2}} \frac{\sin^4(x)}{\cos^4(x)}\, dx$$

$$\int_0^{\frac{\pi}{2}} \tan^4(x)\, dx \qquad \int_0^{\frac{\pi}{2}} \cot^{-4}(x)\, dx \qquad \int_0^1 32 \frac{z^4}{(1+z^2)\,(1-z^2)^4}\, dx$$

Slagle's Method

Algorithmic transformation (always applied)

- Factor constant

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

- Decomposition

$$\int_a^b \sum f_i(x)dx = \sum \int_a^b f_i(x)dx$$

- Linear substitution

$$\int_a^b f(c_1 + c_2 v)dv = \int_{c_1+c_2 a}^{c_1+c_2 b} \frac{1}{c_2} f(u)du$$

Slagle's Method

Heuristic Transformation (plausible but not always appropriate)

- Substitute a non-linear subexpression whose derivative divides the integrand:

$$\int_a^b x e^{x^2} dx = \int_c^d \frac{1}{2} du \ (\text{substitute } e^{x^2} \text{ by } u, \text{since } \left(e^{x^2}\right)' = 2xe^{x^2})$$

- For each quadratic subexpression of the form $c_2 + c_1 x^2$
  - if $c_1 < 0, c_2 > 0$, try the substitution $\mathrm{x} = \sqrt{\frac{c_2}{-c_1}} \sin(u)$, which replaces the quadratic to $c_2 \cos^2(u)$

$$\int_a^b \frac{x^4}{(1 - x^2)^{5/2}} dx = \int_c^d \frac{\sin^4(x)}{\cos^4(x)} dx$$

  - ......
- ...... (ten rules in total)

Proof Reconstruction

- From a sequence of computation steps, automatically reconstruct proof in higher-order logic. This is possible since all necessary information is already available.

- Main tasks:
  - Proofs for simplification of expressions.
  - Proofs for inequality checking.
  - Applying integration theorems, including check side conditions (e.g. continuity, integrability, …)

- Implementation in HolPy, an interactive theorem prover written in Python, with Python API for proof automation.

Proof Automation

Simplification: reduce expressions to canonical form.

- Each monomial can be converted to the form

$$c\cdot\left(a_1^{p_1}a_2^{p_2}\ldots a_k^{p_k}\right)$$

  - ➢ c: prime number
  - ➢ a_i: prime number / a term whose head is not an arithmetic operator
  - ➢ $p_i \in (0,1)$ if a_i is a prime number
- A polynomial is a sum of monomials which are all distinct and in sorted order.

Example

$$\sqrt{6}\sqrt{2}\left(x+3^{2/3}\right)=6^{1/2}2^{1/2}x+6^{1/2}2^{1/2}3^{2/3}x=2^{1/2}3^{1/2}2^{1/2}x+2^{1/2}2^{1/3}2^{1/2}3^{2/3}x=\underset{c}{2}\cdot\underset{a_1}{3}^{\overset{p_1}{1/2}}\underset{a_2}{x}+6\cdot 3^{1/6}$$

Proof Automation

Inequality checking

A major task in proof automation is inequality checking, for instance, if we want to simplify $\sqrt{f(x)^2}$ to $f(x)$ when $x \in (a, b)$, then we need to prove that

$$x \in (a, b) \Longrightarrow f(x) \geq 0$$

We implemented a heuristic procedure for inequality checking which can be considered as a simplified version of interval arithmetic. For instance, we can deduce $2 - 2 \cdot \sin^2(x) \geq 0$ by the following steps:

$$x \in \left(0, \frac{\pi}{2}\right) \Longrightarrow \sin(x) \in (0,1) \Longrightarrow \sin^2(x) \in (0,1) \Longrightarrow$$
$$2 - 2 \cdot \sin^2(x) \in (0,2) \Longrightarrow 2 - 2 \cdot \sin^2(x) \geq 0$$

In the future, consider other heuristic extensions (e.g. Avigad et al, JAR '16).

Proof Automation

Applying theorems

- Automatic proofs of continuity, integrability, etc. (Currently can only handle the basic cases).
- Use of integration theorems such as substitution, integration by parts (simple because the parameters of the rule already contain required instantiations).

Background library

- Ported statements of over 1000 theorems from HOL Light.
- About 40% are proved using the point-and-click user interface in HolPy (ICFEM ’19).

Experiments

We evaluated our prototype on problems taken from

- Exam preparation books (Tongji)
- Online problems listed by D. Kouba
- MIT Integration Bee

Problem set	Total	Solved	Ratio	Slagle	Ratio	Maple	WolframAlpha
Tongji	36	36	100%	26	72%	32	35
Kouba/Substitution	18	17	94%	13	72%	18	18
Kouba/Exponentials	12	7	58%	7	58%	12	11
Kouba/Trigonometric	27	22	81%	11	41%	18	22
Kouba/ByParts	23	22	96%	17	74%	23	23
Kouba/LogArcTangent	22	21	95%	13	59%	21	21
Kouba/PartialFraction	20	16	80%	8	40%	18	20
MIT/2013	25	20	80%	14	56%	20	24
Total	183	161	88%	109	60%	162	174

Future Work

- **Long-term plan:** implement a symbolic computation tool with correctness guaranteed by generating proofs.

- Extend expressions and integration rules to support:
  - Multivariable integrals;
  - Improper integrals, Laplace and Fourier transforms;
  - Vector, matrix and tensor calculus.

- Generate proofs to different interactive theorem provers:
  - Isabelle, HOL Light, HOL4 (higher-order logic).
  - Coq, Lean (dependent type theory).

Q&A

THANK YOU!