A *t*-test for synthetic controls[*]

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Abstract

We propose a practical and robust method for making inferences on average treatment effects estimated by synthetic controls. We develop a K-fold cross-fitting procedure for bias correction. To avoid the difficult estimation of the long-run variance, inference is based on a self-normalized t-statistic, which has an asymptotically pivotal t-distribution. Our t-test is easy to implement, provably robust against misspecification, and valid with stationary and non-stationary data. It demonstrates an excellent small sample performance in application-based simulations and performs well relative to alternative methods. We illustrate the usefulness of the t-test by revisiting the effect of carbon taxes on emissions.

Keywords: bias correction, cross-fitting, self-normalization, average treatment effect

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1 Introduction

We propose a simple and robust t-test for making inferences on average effects estimated using synthetic control (SC) (Abadie and Gardeazabal, 2003; Abadie et al., 2010; Abadie, 2021). The t-test is easy to implement. For example, one can obtain $(1 - \alpha)$-confidence intervals as

$$\texttt{estimated effect} \pm (1 - \alpha/2) \texttt{ quantile of } t\text{-}\texttt{distribution} \times \texttt{standard error.} \quad (1)$$

We consider a setting with one treated unit, which is untreated for the first T_0 periods and treated for the remaining T_1 periods, and N untreated control units. Let $Y_{0t}(1)$ and $Y_{0t}(0)$ denote the (random) potential outcomes of the treated unit with and without the treatment. Our object of interest is the average treatment effect on the treated unit (ATT) in the post treatment period,

$$\tau = \frac{1}{T_1} \sum_{t=T_0+1}^{T_0+T_1} (Y_{0t}(1) - Y_{0t}(0)) \,.$$

The ATT provides an interpretable and easy-to-communicate summary measure of the effect of the treatment. Average treatment effects on the treated units are popular target parameters in many causal inference problems because of their direct policy relevance. Like much of the SC literature, we focus on settings with only one treated unit and many treated periods. In such settings, the average effect for the treated unit over time captures the impact of the treatment rather than the average effect across many units. We also show how to modify the proposed t-test to make inferences on the expected effect for the treated unit, $E(Y_{0t}(1) - Y_{0t}(0))$, when researchers are willing to restrict treatment effect heterogeneity by imposing stationarity and weak dependence of the treatment effect sequence.

Many existing inference methods for SC with a single treated unit focus on sharp null hypotheses, such as the null hypothesis of no effect whatsoever, and per-period effects (e.g., Abadie et al., 2010; Firpo and Possebom, 2018; Cattaneo et al., 2021; Chernozhukov et al., 2021). As pointed out by Imbens and Rubin (2015, p.81), sharp null hypotheses are "a very useful starting point, prior to any more sophisticated analysis," but rejecting sharp null hypotheses is "not sufficient to inform policy decisions." Per-period effects provide useful information about the effect dynamics. But while we can make inferences about per-period effects, they cannot be consistently estimated when there is only one treated unit, and the resulting confidence intervals can be wide and uninformative. Moreover, reporting inferences on many per-period effects might not be an effective way to communicate the overall impact of the treatment when there are many treated periods, and often an interpretable one-number summary is called for. The ATT provides such a summary, is a natural inferential

target empirical work, and admits confidence intervals that take the "standard" form (1). By contrast, the existing methods for inference on sharp nulls and per-period effects are typically based on permutation or bootstrap approaches.

The main inferential challenge in SC applications is that $Y_{0t}(0)$ is unobserved for $t > T_0$. SC approximates this unobserved counterfactual using a weighted average of control outcomes, $\sum_{i=1}^{N} \hat{w}_i Y_{it}$. The weights $(\hat{w}_1, \ldots, \hat{w}_N)$ are estimated based on the pre-treatment data and constrained to be positive and add up to one.

The natural SC estimator of the ATT is

$$\hat{\tau}^{\mathrm{SC}} = \frac{1}{T_1} \sum_{t=T_0+1}^{T_0+T_1} \left(Y_{0t} - \sum_{i=1}^{N} \hat{w}_i Y_{it} \right). \tag{2}$$

In many SC applications, T_0 is smaller than or comparable to N, and the data exhibit persistence and dynamics. In such settings, there are two major challenges when using $\hat{\tau}^{\mathrm{SC}}$ to make inferences about τ. First, $\hat{\tau}^{\mathrm{SC}}$ is biased due the error from estimating the high-dimensional (relative to T_0) weights, even under correct specification, and the bias can be substantial under misspecification (Figure 1).[1] This bias of $\hat{\tau}^{\mathrm{SC}}$ precludes the application of standard inference procedures.

Second, even in the ideal case where the true SC weights are known, estimating standard errors is difficult since they depend on the long-run variance (LRV). It is well-known that classical LRV estimators (e.g., Newey and West, 1987; Andrews, 1991) are not accurate enough for reliable inference in small samples. Figure 2 shows that using the popular Newey-West standard errors yields substantial undercoverage, especially when T_0 is small.[2]

We develop an inference method that addresses both challenges and is motivated by an asymptotic framework where $T_0, T_1, N \to \infty$. To remove the bias of $\hat{\tau}^{\mathrm{SC}}$, we propose a K-fold cross-fitting procedure. To avoid the difficult LRV estimation, we propose a self-normalized t-statistic with an asymptotically pivotal t-distribution with $K - 1$ degrees of freedom, which makes our method easy to implement. Self-normalization further induces theoretical higher-order improvements and yields excellent small sample properties in our simulations. Such higher-order improvements and excellent small sample properties are important for a method relying on asymptotic approximations since T_0 and T_1 are often not very large in SC applications. Our method is implemented in the R-package scinference (https://github.com/kwuthrich/scinference).

Figure 1 shows that the distribution of the bias-corrected estimator is centered at the true value ($\tau = 0$) and well-approximated by a normal distribution, even under misspecification.

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[1] In our context, correct specification refers to settings where the population SC estimator is unbiased.

[2] As pointed out in Müller (2007), any consistent estimator of LRV might suffer from robustness issues.

Figure 2 shows that our t-test exhibits an excellent coverage accuracy, despite the small sample sizes.

Figure 1: Bias of synthetic control and impact of debiasing
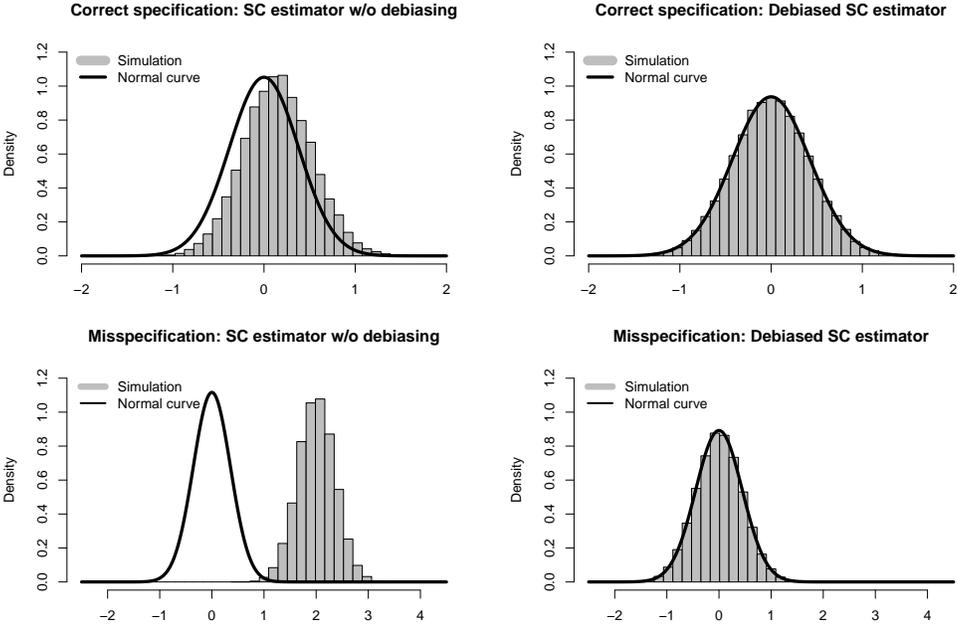
Notes: Simulations with 50,000 repetitions. $Y_{0t}(0) = \mu + \sum_{i=1}^{N} w_i Y_{it}(0) + u_t$, $Y_{it}(0) = 2 \cdot 1\{i \leq 3\} + v_{it}, v_{it} \overset{iid}{\sim} N(0,1)$, $\{u_t\}$ is a Gaussian AR(1) process with coefficient 0.31, and $(T_0, T_1, N) = (30, 16, 14)$ as in the empirical application. Correct specification: $(\mu, w) = (0, (1/3, 1/3, 1/3, 0 \ldots, 0)')$. Misspecification: $(\mu, w) = (2, (1/3, 1/3, 1/3, 0 \ldots, 0)')$. $Y_{0t}(1) - Y_{0t}(0) = 0$ for $t > T_0$. Debiasing based on $K = 3$.

The t-test is valid with stationary and non-stationary data. As a result, researchers do not need to pre-test for nonstationarity and choose the inference method according to the pre-test. This is important since nonstationarity is a common feature of SC applications. Existing methods typically rely on assumptions on the type of nonstationarity (e.g., unit root or deterministic trend) (e.g., Li, 2020). However, it is well-known that even the slightest misspecification in modeling nonstationarity (e.g., unit root vs. near unit root) yields invalid inferences. Therefore, we take a different approach and do not make assumptions about the exact nature of the nonstationarity. Instead, we restrict the heterogeneity in the nonstationarity across units. That is, we require the units to be sufficiently similar, which is a crucial contextual requirement in SC studies (Abadie, 2021). Avoiding assumptions about the exact type of nonstationarity is important because one of the main reasons for using SC is that the precise nature of the counterfactual trend is unknown.

With stationary data, the t-test is valid under under arbitrary misspecification. With nonstationary data, we establish the validity of the t-test under two different settings. First, the

Figure 2: Undercoverage with Newey-West standard errors
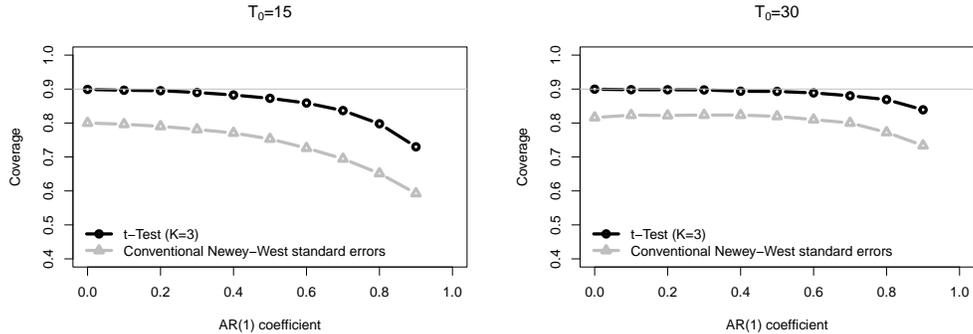
Notes: Simulations with 50,000 repetitions. Nominal coverage: 90%. $Y_{0t}(0) = \sum_{i=1}^{N} w_i Y_{it}(0) + u_t$, $\{u_t\}$ is a Gaussian AR(1) process, $Y_{0t}(1) - Y_{0t}(0) = 0$ for $t > T_0$, and $(T_1, N) = (16, 14)$ as in the empirical application. We focus on the Oracle case where w is known to abstract from the estimation error in $\hat{w}$. The LRV is estimated based on the pre-treatment data using the `NeweyWest` command (`R`-package `sandwich` (Zeileis, 2004; Zeileis et al., 2020)) with pre-whitening.

t-test is valid under arbitrary misspecification when all units share a common nonstationarity. Second, the t-test remains valid when units deviate from the common nonstationarity under restrictions on the magnitude and heterogeneity of the deviations. The second result covers many relevant types of nonstationarity, such as heterogeneous deterministic time trends and certain forms of cointegration, but requires SC to be correctly specified.

A by-product of our method is improved asymptotic efficiency over difference-in-differences (DID). We show that the asymptotic variance of the bias-corrected SC estimator is no larger than that of DID, irrespective of whether the SC model is correctly specified or not. Moreover, the t-test is valid when the common trends assumption underlying DID is violated and thus also more robust than DID.

We illustrate the usefulness of our method by revisiting Andersson (2019a)'s analysis of the effect of a carbon tax on emissions between 1990 and 2005 in Sweden. In this application, the ATT captures the average effect of the carbon tax on emissions — a one-number summary of the overall effect that Andersson (2019a, p.14) explicitly mentions. The t-test provides robust confidence intervals for this parameter. The estimated ATT is negative and significant. Our findings complement and corroborate the inference results in Andersson (2019a), which are based on the permutation test of Abadie et al. (2010).

The t-test demonstrates an excellent small sample performance in simulations calibrated based on the empirical application and performs well relative to existing alternatives such as DID, subsampling (Li, 2020), and synthetic DID (SDID) (Arkhangelsky et al., 2021).

Finally, we provide recommendations for practice. We discuss when to use the t-test and what to consider when implementing it in applications.

1.1 Related literature

We contribute to the literature by proposing methods for making inferences on the ATT and expected treatment effects based on SC in settings with one or few treated units. The closest papers to ours are Li (2020) and Arkhangelsky et al. (2021). Here we compare and contrast the proposed t-test with their contributions. See Section 5 for a simulation comparison and Section 7 for guidance on which method to choose in applications.

Li (2020) proposes a subsampling approach to inference in SC settings with a small number of controls (N is assumed to be fixed).[3] They derive the asymptotic distribution of the SC estimator of $E(\alpha_t)$ using a projection framework with stationary, trend-stationary, and unit root data. For inference, they propose a subsampling procedure. They note that $\sqrt{T_1}(\hat{\tau}^{\mathrm{SC}} - E(\alpha_t))$ can be decomposed into two parts, A_1 and A_2. Since only A_1 depends on the estimated SC weights, subsampling is only required for this part, and A_2 can be approximated using the bootstrap. Li (2020) establishes the validity of this procedure for the case where the SC prediction errors and $\{\alpha_t - E(\alpha_t)\}$ are serially uncorrelated. When the data are trend-stationary, they recommend detrending before applying the inference method.

Our t-test differs from Li (2020) in several important aspects. We allow for a large number of control units (N can grow with (T_0, T_1)) and explicitly correct for the bias of SC using a cross-fitting approach. This leads to a self-normalized t-statistic and an inference procedure with higher-order improvements that does not require pre-processing the data to make them stationary. The t-test also avoids subsampling, which may not perform well in small samples.

Arkhangelsky et al. (2021) propose synthetic DID (SDID), which combines ideas from DID and SC. In our notation, their estimator of the ATT, $\hat{\tau}^{\mathrm{SDID}}$, is given by

$$(\hat{\tau}^{\mathrm{SDID}}, \hat{\mu}, \hat{\alpha}, \hat{\beta}) = \arg\min_{\tau, \mu, \alpha, \beta} \sum_{i=0}^{N} \sum_{t=1}^{T_0+T_1} (Y_{it} - \mu - \alpha_i - \beta_t - D_{it}\tau)^2 \hat{w}_i^{\mathrm{SDID}} \hat{\lambda}_t^{\mathrm{SDID}}, \qquad (3)$$

where D_{it} is the treatment indicator, $\hat{w}^{\mathrm{SDID}}$ are unit weights that balance the pre-treatment trends of treated and control units, and $\hat{\lambda}^{\mathrm{SDID}}$ are time weights that balance the pre- and post-treatment periods. SDID thus adds time weights and unit fixed effect to the SC method. Both of these additions help reduce the bias of standard SC and improve robustness. Arkhangelsky et al. (2021) establish consistency and asymptotic normality of SDID under a latent factor model for the potential outcomes. To make inferences with one treated unit, they build on ideas in Conley and Taber (2011) to propose a placebo variance estimator

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[3]Here we describe Li (2020)'s method in the context of the SC estimator $\hat{\tau}^{\mathrm{SC}}$. We note that their theory also covers more general SC estimators, such as SC estimators without the adding-up constraint.

that relies on homoskedasticity across units.[4]

Our t-test differs from SDID with respect to the underlying model, the debiasing strategy, and the inference method. The t-test is developed based on a linear prediction model, which can be motivated by but does not rely on factor models. We use a cross-fitting approach that directly removes the bias if the bias is stable over time. To make inferences, we exploit stationarity over time and build on the cross-fitting structure to construct a self-normalized t-statistic that avoids estimating the LRV, is theoretically valid across a variety of settings, and enjoys higher-order improvements.

Throughout the paper, we focus on SC estimators of the counterfactual. However, our method only requires ℓ_2-consistency of the estimator of the weights, which can be established for many penalized regression estimators. Therefore, our paper further contributes to the literature on inference methods based on standard and penalized regression estimators motivated by asymptotics where $T_0, T_1 \to \infty$. Building on the framework of Hsiao et al. (2012), Li and Bell (2017) propose a least squares method for making inferences on the expected treatment effect based on estimators of the LRV when the data are stationary. Carvalho et al. (2018) propose a Lasso-based method for making inferences on the ATT based on LRV estimators under sparsity when the data are stationary.[5] Masini and Medeiros (2020) study the asymptotic distribution of counterfactual estimators based on least squares when the data are non-stationary. They show that the limiting distribution is generally nonstandard and depends on T_0/T_1. They propose a subsampling method for inference when $T_0 \approx T_1$. Compared to this strand of the literature, our t-test is generic in that it accommodates a wide range of penalized and unpenalized regression estimators, is valid under general forms of nonstationarity, does not rely on estimating the LRV, and enjoys higher-order improvements. Moreover, the debiased ATT estimator is asymptotically normal across all settings we consider, which, together with the self-normalized t-statistic, makes the proposed method easy to implement.

Focusing on average effects over time and expected effects, we complement the existing procedures for testing sharp null hypotheses about the treatment effect trajectory $\{Y_{0t}(1) - Y_{0t}(0)\}_{t=T_0+1}^{T_0+T_1}$ and making inferences on per-period effects (e.g., Abadie et al., 2010; Firpo and Possebom, 2018; Cattaneo et al., 2021; Ben-Michael et al., 2021; Chernozhukov et al., 2021; Masini and Medeiros, 2021; Shaikh and Toulis, 2021).[6] Within that strand of literature, our

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[4]An appealing feature of SDID is that it naturally accommodates multiple treated units. For such settings, Arkhangelsky et al. (2021) propose a bootstrap and a jackknife method for inference.

[5]In Section 4.1, they consider an extension to trending regressors but do not provide inference methods for high-dimensional settings.

[6]The time series permutation approach of Chernozhukov et al. (2021) can be extended to test hypotheses about the ATT by collapsing the data across time, provided that $T_0 \gg T_1$. More recently, Cattaneo et al.

paper is most closely related to Ben-Michael et al. (2021), given our focus on bias correction. Ben-Michael et al. (2021) propose a debiasing approach based on an outcome model and suggest using a conformal prediction approach for making inferences on per-period effects, building on Chernozhukov et al. (2021). Finally, it is worthwhile noting that the t-test relies on a sampling-based framework in which the potential outcomes are viewed as random, whereas some of the approaches for testing sharp null hypotheses (e.g., Abadie et al., 2010; Firpo and Possebom, 2018) are motivated from a design-based perspective.

1.2 Notation

Let $\mathbf{1}_p$ denote a $p \times 1$ dimensional vector of ones. For $q \geq 1$, we denote the ℓ_q-norm of a vector as $\| \cdot \|_q$. For a matrix A, we denote $\|A\|_\infty = \|\text{vec}(A)\|_\infty$, where $\text{vec}(A)$ is the column-wise vectorization of A. We write $a \lesssim b$ to denote $a \leq cb$ for some constant $c > 0$ that does not depend on the sample size. We write $a \asymp b$ to denote $a \lesssim b$ and $b \lesssim a$. For a set A, $|A|$ is the cardinality of A.

2 A t-test for synthetic controls

2.1 Setup and object of interest

We consider a synthetic control setup with one treated unit, N control units, and T periods (e.g., Abadie et al., 2010; Doudchenko and Imbens, 2016; Kellogg et al., 2021). The treated unit is untreated for the first T_0 periods and treated for the remaining $T - T_0 = T_1$ periods. The control units remain untreated throughout. The potential outcomes with and without the treatment are $Y_{it}(1)$ and $Y_{it}(0)$, respectively. In the main text, we assume that the treatment status is fixed and label the treated unit as $i = 0$ and the control units as $i = 1, \ldots, N$. In Appendix A, we provide sufficient conditions under which the t-test remains valid when the treatment status is random and selection is based on past outcomes or latent variables in factor models. Observed outcomes are given by $Y_{it} = Y_{it}(0) + \alpha_{it} \mathbf{1}\{i = 0, t > T_0\}$, where $\alpha_{it} := Y_{it}(1) - Y_{it}(0)$ is the treatment effect for unit i in period t.

The existing inference approaches for SC and related methods with one (or few) treated units differ regarding the identification assumptions they rely on and what is assumed to be random and fixed, respectively. We consider a setting where the potential outcomes are random and the treatment effect sequence can be either random or fixed.[7]

(2023) extended the method in Cattaneo et al. (2021) to accommodate more general treatment effects, including ATT over time and across units in staggered adoption designs.

[7]Similar settings are considered by Li (2020), Arkhangelsky et al. (2021), Ben-Michael et al. (2021),

Our main goals are to estimate and construct confidence intervals for the ATT,

$$\tau = \frac{1}{T_1} \sum_{t=T_0+1}^{T} \alpha_{0t}. \tag{4}$$

When $\{\alpha_{0t}\}$ and thus τ are random, these confidence intervals are not confidence intervals in the traditional sense, but should be interpreted as prediction intervals (Cattaneo et al., 2021; Chernozhukov et al., 2021; Cattaneo et al., 2023). See Sections 2.2 and 3.1 for further discussions. To simplify the exposition, we sometimes suppress the subscript "0" and write $\alpha_t = \alpha_{0t}$ whenever there is no ambiguity.

The ATT is an interpretable and easy-to-communicate summary measure of the impact of the treatment when treatment effects are heterogeneous over time. For example, in our reanalysis of Andersson (2019a), the ATT captures the average per-period effect of a carbon tax on emissions in Sweden between 1990, when the tax was introduced, and 2005, when the EU started its emissions trading system. Andersson (2019a, p.14) explicitly mentions the ATT when discussing the empirical results. In settings with only one treated unit, the ATT in (4) is equivalent, for example, to the treatment effects considered by Carvalho et al. (2018), Arkhangelsky et al. (2021), and Cattaneo et al. (2023).

Remark 1. *The proposed method can be applied to make inferences on the ATT over subperiods of the post-treatment period, $\mathcal{T}^s = \{T_0+1+r, \ldots, T-s\}$, where $r, s \geq 0$, $|\mathcal{T}^s|^{-1} \sum_{t \in \mathcal{T}^s} \alpha_t$. Our asymptotic theory requires that $|\mathcal{T}^s| \to \infty$, so that $|\mathcal{T}^s|$ needs to be large enough for the asymptotic approximations to be accurate.* □

Remark 2. *In settings where researchers are willing to restrict treatment effect heterogeneity over time and assume that the treatment effect sequence $\{\alpha_t\}$ is stationary and weakly dependent, the expected treatment effect, $E(\alpha_t)$, is a natural alternative to the ATT (e.g., Li and Bell, 2017; Li, 2020). We propose a modification of our method for making inferences on $E(\alpha_t)$ in Appendix B.3.* □

2.2 Implementing the t-test

To remove the bias of SC, we employ a K-fold cross-fitting procedure, where K is fixed. We discuss the choice of K in Section 3.2. We choose K consecutive blocks from the pre-treatment period: $H_1 \bigcup H_2 \bigcup \cdots \bigcup H_K \subseteq \{1, \ldots, T_0\}$. Define $r = \min\{\lfloor T_0/K \rfloor, T_1\}$ and let

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Cattaneo et al. (2021), Chernozhukov et al. (2021), Ferman (2021), Ferman and Pinto (2021), and Cattaneo et al. (2023). An alternative and complementary strand of the literature focuses on design-based inference, treating the potential outcomes as fixed and leveraging assumptions on the assignment process (e.g., Abadie et al., 2010; Firpo and Possebom, 2018).

$H_k = \{(k-1)r+1, \ldots, kr\}$ for $1 \leq k \leq K$.[8] For simplicity, we assume that T_0/K is an integer. For $k = 1, \ldots, K$, compute

$$\hat{\tau}_k = \frac{1}{T_1} \sum_{t=T_0+1}^{T} \left(Y_{0t} - \sum_{i=1}^{N} \hat{w}_{i,(k)} Y_{it} \right) - \frac{1}{|H_k|} \sum_{t \in H_k} \left(Y_{0t} - \sum_{i=1}^{N} \hat{w}_{i,(k)} Y_{it} \right), \tag{5}$$

where $\hat{w}_{(k)} = (\hat{w}_{1,(k)}, \ldots, \hat{w}_{N,(k)})'$ is obtained by applying SC to the data in $H_{(-k)} := \{1, \ldots, T_0\} \setminus H_k$. This construction ensures that under weak dependence, $\hat{w}_{(k)}$ is approximately independent of the data in $H_k \bigcup \{T_0 + 1, \ldots, T\}$, which allows us to establish the theoretical validity of our procedure under weak conditions.

The basic idea behind the construction of estimator (5) is as follows. The first component, $T_1^{-1} \sum_{t=T_0+1}^{T} \left(Y_{0t} - \sum_{i=1}^{N} \hat{w}_{i,(k)} Y_{it} \right)$, corresponds to the natural SC estimator $\hat{\tau}^{\mathrm{SC}}$ in (2) with $\hat{w}$ replaced by $\hat{w}_{(k)}$. This estimator is biased (see Figure 1). The second component, $|H_k|^{-1} \sum_{t \in H_k} \left(Y_{0t} - \sum_{i=1}^{N} \hat{w}_{i,(k)} Y_{it} \right)$, is an estimator of the bias of this estimator in the pre-treatment period. Under the assumptions specified below, the bias is the same in the pre-treatment and the post-treatment period. As a result, subtracting the second component removes the bias.

For concreteness, we consider the following canonical SC estimator (e.g., Doudchenko and Imbens, 2016) in the main text:

$$\hat{w}_{(k)} \in \arg\min_{w \in \mathcal{W}^{SC}} \sum_{t \in H_{(-k)}} \left(Y_{0t} - \sum_{i=1}^{N} w_i Y_{it} \right)^2, \tag{6}$$

where $\mathcal{W}^{SC} := \left\{ w : w_i \geq 0, \sum_{i=1}^{N} w_i = 1 \right\}$.[9] We study the theoretical properties of the classical SC estimator of Abadie et al. (2010) in Appendix B.2. The t-test only requires an ℓ_2-consistent estimator of the weights. Therefore, it also works in conjunction with many other SC estimators and penalized regression estimators; see Remark 4.

The final estimator of the ATT is simply the average of $\hat{\tau}_1, \ldots, \hat{\tau}_K$,

$$\hat{\tau} = \frac{1}{K} \sum_{k=1}^{K} \hat{\tau}_k. \tag{7}$$

To avoid the difficult estimation of the LRV, we construct a scale-free test statistic. The idea is to form a ratio in which the numerator and the denominator are both scaled by the long-run standard deviation. Specifically, we construct a quantity based on $\hat{\tau}_1, \ldots, \hat{\tau}_K$:

$$\mathbb{T}_K = \frac{\sqrt{K} (\hat{\tau} - \tau)}{\hat{\sigma}_{\hat{\tau}}}, \tag{8}$$

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[8]Here, we choose to use the first K blocks. Other choices such as the last K blocks are also valid.

[9]The argmin may not be unique if $T_0 - r < N$. In this case, we can take $\hat{w}_{(k)}$ to be any element in the argmin set.

where

$$\hat{\sigma}_{\hat{\tau}} = \sqrt{1 + \frac{Kr}{T_1}} \sqrt{\frac{1}{K-1} \sum_{k=1}^{K} (\hat{\tau}_k - \hat{\tau})^2}.$$

In Sections 3 and 4 we show that $\mathbb{T}_K$ has an asymptotic t-distribution with $K-1$ degrees of freedom as $T_0, T_1 \to \infty$. Note that $\hat{\sigma}_{\hat{\tau}}/\sqrt{K}$ is different from standard errors based on asymptotic normality and consistent estimators of the variance. In our framework $\hat{\sigma}_{\hat{\tau}}$ is a random variable since K is fixed, as in the classical statistical analysis of t-tests based on a fixed number of Gaussian variables.

The t-statistic (8) differs from the standard t-statistic by the factor $\sqrt{1 + Kr/T_1}$ in the denominator. This rescaling is necessary in our context because the $\hat{\tau}_1, \ldots, \hat{\tau}_K$ are not asymptotically independent since they share a common component coming from the average over the post-treatment period. The common component cancels out in the denominator but not in the numerator of $\mathbb{T}_K$. Therefore, to account for this common component and show that the test statistic $\mathbb{T}_K$ has an asymptotic t-distribution, we need to divide the numerator by $\sqrt{1 + Kr/T_1}$. We refer to the proof of Theorem 2 for a formal discussion.

The asymptotic t-distribution of $\mathbb{T}_K$ suggests the following $(1-\alpha)$ confidence interval for τ:

$$\mathcal{I}_K(1-\alpha) = \left[\hat{\tau} - t_{K-1}(1-\alpha/2)\frac{\hat{\sigma}_{\hat{\tau}}}{\sqrt{K}}, \ \hat{\tau} + t_{K-1}(1-\alpha/2)\frac{\hat{\sigma}_{\hat{\tau}}}{\sqrt{K}} \right], \tag{9}$$

where $t_{K-1}(1-\alpha/2)$ is the $(1-\alpha/2)$-quantile of a student t-distribution with $K-1$ degrees of freedom. The interpretation of $\mathcal{I}_K(1-\alpha)$ depends on whether τ is fixed or random. If τ is fixed, $\mathcal{I}_K(1-\alpha)$ can be interpreted as a conventional confidence interval; if τ is random, $\mathcal{I}_K(1-\alpha)$ can be interpreted as a prediction interval (as in Cattaneo et al., 2021; Chernozhukov et al., 2021; Cattaneo et al., 2023). For simplicity, we will refer to $\mathcal{I}_K(1-\alpha)$ as a confidence interval throughout.

Remark 3. *The construction of $\mathbb{T}_K$ is related to Ibragimov and Müller (2010) with two important differences. First, $\{\hat{\tau}_k\}$ naturally arise from our cross-fitting procedure for bias correction. Second, the $\hat{\tau}_k$s share a common component and, consequently, are not asymptotically independent (Theorem 1). Nevertheless, we are able to show that, after scaling the denominator by $\sqrt{1 + Kr/T_1}$, $\mathbb{T}_K$ has an asymptotic t-distribution under the null hypothesis (Theorem 2). Our construction of $\mathbb{T}_K$ is also related to the Fama and MacBeth (1973) variance estimator (e.g., Cattaneo et al., 2020).* □

3 Theoretical properties with stationary data

Here, we show that our t-test is valid with stationary data, robust to misspecification, and more efficient than DID. We consider inference in a repeated sampling framework under asymptotics where $T_0, T_1 \to \infty$, and our results also allow for $N \to \infty$.

3.1 Asymptotic theory

Following the SC literature, we predict $Y_{0t}(0)$ for $t > T_0$ using a linear combination of $(Y_{1t}(0), \ldots, Y_{Nt}(0))$. Defining $Y_t(0) := Y_{0t}(0)$, $Y_t := Y_{0t}$, and $X_t := (Y_{1t}(0), \ldots, Y_{Nt}(0))'$, the linear prediction model can be written as

$$Y_t(0) = X_t'w_* + u_t, \quad 1 \le t \le T, \tag{10}$$

where the pseudo-true SC weights are defined as $w_* := \arg\min_{w \in \mathcal{W}^{SC}} E(Y_t(0) - X_t'w)^2$ and the pseudo-true residuals or prediction errors are $u_t := Y_t(0) - X_t'w_*$.[10] We interpret model (10) as a statistical or predictive model and not as a structural model.[11] A major advantage of this interpretation is that it allows SC to be misspecified and the pseudo-true weights w_* to be different from the true (infeasible) SC weights.[12] Allowing for misspecification is important in practice. For example, under a linear factor model for the potential outcomes, SC does not recover the true weights and is biased in general (e.g., Ferman and Pinto, 2021). More generally, the true model might be nonlinear. The proposed inference method is valid in both of these cases and also accommodates many other forms of misspecification. In the main text, we assume that the predictive relationship is stable over time. In Appendix B.1, we show that the t-test can accommodate certain forms of time-varying weights.

Throughout this section, we maintain the following standard stationarity assumption. We establish the properties of our method with non-stationary data in Section 4.

Assumption 1. $\{(Y_t(0), X_t)\}_{t=1}^T$ *is covariance-stationary.*

By simple algebra, Assumption 1 gives us the following observation.

Lemma 1. *Let Assumption 1 hold. Then, for $1 \le k \le K$,*

$$\hat{\tau}_k - \tau = \left(\frac{1}{T_1} \sum_{t=T_0+1}^{T} u_t - \frac{1}{|H_k|} \sum_{t \in H_k} u_t \right) + \left(\frac{1}{|H_k|} \sum_{t \in H_k} \tilde{X}_t - \frac{1}{T_1} \sum_{t=T_0+1}^{T} \tilde{X}_t \right)' \Delta_{(k)},$$

---

[10]If the best linear predictor is well-defined and satisfies the SC constraints so that $w_* = [E(X_tX_t')]^{-1}E(X_tY_t(0)) \in \mathcal{W}^{SC}$, then $E(X_tu_t) = 0$. However, in practice, $\mathcal{W}^{SC}$ could be "too small" such that $[E(X_tX_t')]^{-1}E(X_tY_t(0)) \notin \mathcal{W}^{SC}$ and $E(X_tu_t) \neq 0$.

[11]See, for example, Cattaneo et al. (2021) and Chernozhukov et al. (2021) for similar interpretations.

[12]Starting with the seminal paper by Abadie et al. (2010), the true SC weights have often been defined via a factor model for the potential outcomes.

where $\tilde{X}_t = X_t - E(X_t)$ and $\Delta_{(k)} = \hat{w}_{(k)} - w_*$.

To establish the asymptotic properties of our method, we need to show that the second term in Lemma 1 is negligible. For that, we impose additional assumptions.

We start by imposing ℓ_2-consistency of the SC estimator $\hat{w}_{(k)}$.

Assumption 2. $\max_{1 \leq k \leq K} \|\hat{w}_{(k)} - w_*\|_2 = o_P(1)$.

We state Assumption 2 in terms of the pseudo-true SC weights $w_* := \arg\min_{w \in \mathcal{W}^{SC}} E(Y_t(0) - X_t'w)^2$. The results in Theorems 1 and 2 below continue to hold as long as the estimators $\hat{w}_{(k)}$ converge to any time-invariant vector of weights.

The following lemma verifies Assumption 2 for the SC estimator $\hat{w}_{(k)}$. It allows N to be large relative to T_0.

Lemma 2. *Suppose Assumption 1 and the following conditions hold.*

1. $\max_{1 \leq k \leq K} \|\hat{\mu}_{(-k)} - \mu\|_\infty = o_P(1)$, *where* $\mu = E X_t Y_t(0)$ *and* $\hat{\mu}_{(-k)} = |H_{(-k)}|^{-1} \sum_{t \in H_{(-k)}} X_t Y_t(0)$.

2. $\max_{1 \leq k \leq K} \|\hat{\Sigma}_{(-k)} - \Sigma\|_\infty = o_P(1)$ *and* $\lambda_{\min}(\Sigma) \geq c$, *where* $\Sigma = E X_t X_t'$ *and* $\hat{\Sigma}_{(-k)} = |H_{(-k)}|^{-1} \sum_{t \in H_{(-k)}} X_t X_t'$.

Then, we have that $\max_{1 \leq k \leq K} \|\hat{w}_{(k)} - w_*\|_2 = o_P(1)$. *In particular,*

$$\max_{1 \leq k \leq K} \|\hat{w}_{(k)} - w_*\|_2^2 \leq \frac{8\|\hat{\Sigma}_{(-k)} - \Sigma\|_\infty + 2\|\hat{\mu}_{(-k)} - \mu\|_\infty}{c}.$$

The first condition in Lemma 2 holds under weak serial dependence, mild conditions on the tail of the distribution of the variables, and conditions on N. For example, when the entries of X_t and $Y_t(0)$ are sub-Gaussian, we can allow for $\log N = o(\sqrt{T_0})$; when the entries of X_t and $Y_t(0)$ have bounded qth moment for $q > 2$, then we can typically allow for $N = o(T_0^{q/4})$. This feature is essential because N and T_0 have a similar order of magnitude in many SC applications. The second condition requires the eigenvalues of $\Sigma_{(-k)}$ to be bounded away from zero to achieve identification of the pseudo-true SC weights. We emphasize that Lemma 2 does not impose any sparsity assumptions on the weights.

We also impose weak dependence assumptions on the data. Define $\tilde{u}_t := u_t - E(u_t)$ and $\sigma^2 = \lim_{T \to \infty} E\left(T^{-1/2} \sum_{t=1}^{T} \tilde{u}_t\right)^2$.

Assumption 3. *Suppose the following conditions hold.*

1. *There exists a constant* $\kappa_1 > 0$ *such that for any* $A \subseteq \{1, \ldots, T\}$, *the largest eigenvalue of* $E\left[|A|^{-1} \left(\sum_{t \in A} \tilde{X}_t\right) \left(\sum_{t \in A} \tilde{X}_t\right)'\right]$ *is bounded above by* κ_1.

2. *There exists a sequence $\rho_T > 0$ such that $P(\max_{1 \leq t \leq T} \|\tilde{X}_t\|_\infty \leq \rho_T) \to 1$.*

3. *The data $\{(X_t, \tilde{u}_t)\}_{t=1}^T$ are β-mixing with coefficient satisfying $\beta_{\mathrm{mix}}(\gamma_T) \to 0$ for some sequence γ_T satisfying $0 < \gamma_T < r/2$ and $\rho_T \gamma_T = o_P(\min\{\sqrt{T_0}, \sqrt{T_1}\})$, where $r = \min\{\lfloor T_0/K \rfloor, T_1\}$.*

4. *$\{\tilde{u}_t\}_{t=1}^T$ satisfies $\max_{1 \leq t \leq T} E|\tilde{u}_t|^q = O(1)$ and $\beta_{\mathrm{mix}}(i) \lesssim i^{-\eta}$ for some constants $q > 2$ and $\eta > q/(q-2)$ and $\sigma^2 > 0$.*

The weak dependence is stated in terms of β-mixing, which holds for a large class of stochastic processes. Stationarity and β-mixing conditions are commonly imposed for time series data. These conditions are satisfied in various Markov chains and hidden Markov models, including ARMA, GARCH, and many stochastic volatility models (e.g., Carrasco and Chen, 2002; Meyn and Tweedie, 2012). Note that the β-mixing and moment conditions in Assumption 3 rule out unit-root or near-unit-root processes.

The following theorem establishes the asymptotic distribution of the component estimators. Let $g_{c_0, K} = K\mathbf{1}\{c_0 < 1\} + (K/c_0)\mathbf{1}\{1 \leq c_0 \leq K\} + \mathbf{1}\{c_0 > K\}$.

Theorem 1. *Let Assumptions 1, 2, and 3 hold. Suppose that $T_0, T_1 \to \infty$ and that $T_0/T_1 \to c_0$ for some $c_0 \in [0, \infty]$. Then,*

$$\sqrt{\min\{T_0, T_1\}} \begin{pmatrix} \hat{\tau}_1 - \tau \\ \vdots \\ \hat{\tau}_K - \tau \end{pmatrix} \overset{d}{\to} \begin{pmatrix} \sqrt{\min\{c_0, 1\}}\xi_0 - \sqrt{g_{c_0,K}}\xi_1 \\ \vdots \\ \sqrt{\min\{c_0, 1\}}\xi_0 - \sqrt{g_{c_0,K}}\xi_K \end{pmatrix} \sigma,$$

where $\xi_0, \ldots, \xi_K$ are independent $N(0,1)$ random variables.

The proof of Theorem 1 proceeds in two steps. First, observe that $\hat{w}_{(k)} - w_*$ is approximately independent of the data in $H_k \bigcup \{T_0 + 1, \ldots, T\}$ under weak dependence of the data (Assumption 3). Consequently, ℓ_2-consistency of $\hat{w}_{(k)}$ can be used to bound the second term in Lemma 1, so that

$$\hat{\tau}_k - \tau - \left(\frac{1}{T_1} \sum_{t=T_0+1}^T u_t - \frac{1}{|H_k|} \sum_{t \in H_k} u_t \right) = o_P\left(\frac{1}{\sqrt{\min\{T_0, T_1\}}} \right). \tag{11}$$

Second, under stationarity, we can replace u_t in (11) by $\tilde{u}_t$, which is mean-zero by construction. As a consequence, the desired result follows from a CLT. The common component ξ_0 corresponds to the post-treatment average, and $\xi_1, \ldots, \xi_K$ correspond the averages over the blocks in the pre-treatment period.

Theorem 1 imposes no restrictions on the relative magnitude of T_0 and T_1, accommodating a wide range of applications. This feature adds useful robustness since researchers do not have

to choose between different asymptotic approximations and inference procedures depending on the relative magnitude of T_0 and T_1.

The next theorem establishes the asymptotic distribution of the ATT estimator and $\mathbb{T}_K$.

Theorem 2. *Let the assumptions in Theorem 1 hold. Then (i)*

$$\sqrt{\min\{T_0, T_1\}}(\hat{\tau} - \tau) \xrightarrow{d} N(0, (\min\{c_0, 1\} + g_{c_0, K} K^{-1})\sigma^2)$$

and (ii) $\mathbb{T}_K \xrightarrow{d} t_{K-1}$, *where the random variable* t_{K-1} *has a standard t-distribution with* $K-1$ *degrees of freedom.*

Part (i) of Theorem 2 is a direct consequence of Theorem 1 and establishes the asymptotic normality of the ATT estimator. Making inferences directly based on this result requires estimating the LRV σ^2, which is difficult in small sample settings. We therefore use the self-normalized test statistic $\mathbb{T}_K$, which allows us to avoid estimating the LRV. Part (ii) demonstrates that $\mathbb{T}_K$ has an asymptotically pivotal student t-distribution with $K-1$ degrees of freedom. This result is useful from a practical perspective as one does not have to simulate non-standard critical values, nor rely on subsampling or permutation distributions.

The result in Part (ii) is different from classical results on t-statistics because the component estimators $\hat{\tau}_1, \ldots, \hat{\tau}_K$ are not asymptotically independent due to presence of the common component ξ_0 in the asymptotic distribution in Theorem 1. However, the common component cancels out in the denominator of the t-statistic so that numerator and denominator are asymptotically independent and $\mathbb{T}_K$ has an asymptotic t-distribution after rescaling the denominator to account for the presence of ξ_0 in the numerator.

The following corollary of Theorem 2 formally establishes the $(1-\alpha)$ coverage guarantee of the confidence interval $\mathcal{I}_K(1-\alpha)$ defined in (9).

Corollary 1. *Let the assumptions in Theorem 1 hold. Then,* $P(\tau \in \mathcal{I}_K(1-\alpha)) \to 1 - \alpha$.

In light of Corollary 1, we can interpret $\mathcal{I}_K(1-\alpha)$ as a confidence interval if τ is fixed and as a prediction interval if τ is random. Thus, our framework provides a unified framework, encompassing both leading cases discussed in the literature.

Remark 4. *Theorem 1 only requires* ℓ_2*-consistency of* $\hat{w}_{(k)}$ *(Assumption 2). As a result, the* t*-test is generic and can be used in conjunction with the many other* ℓ_2*-consistent estimators available in the literature (e.g., Lasso). This is similar to the double machine learning (DML) literature (Chernozhukov et al., 2018), where the first-order asymptotics of the final estimator depend on the influence function in the debiasing process, but not on the properties of the estimators of the nuisance parameters, beyond a rate requirement. We emphasize that unlike the DML literature, the t-test does not have any rate requirements.* □

Remark 5. *The alternative fixed-b approach (e.g., Kiefer et al., 2000; Kiefer and Vogelsang, 2002a,b; Jansson, 2004; Kiefer and Vogelsang, 2005; Sun et al., 2008) does not naturally arise from the cross-fitting procedure and encounters extra technical difficulty due to the error from estimating the high-dimensional weights. The usual justification for the fixed-b approach is its higher-order improvement in Gaussian location models. Our procedure achieves the same property in this setting; see Appendix D.* □

Remark 6. *The sign-based randomization test of Canay et al. (2017) is an alternative to the conventional t-test (e.g., Ibragimov and Müller, 2010), provided that K is large enough. In our setting, the randomization test cannot be directly applied due to the dependence between the component estimators $\hat{\tau}_1, \ldots, \hat{\tau}_K$. However, the structure of our problem makes it possible to decorrelate the component estimators without estimating the LRV, so that the randomization test can be applied after decorrelation. Specifically, Theorem 1 shows that $\sqrt{\min\{T_0, T_1\}}(\hat{\tau}_1 - \tau, \ldots, \hat{\tau}_K - \tau)' \xrightarrow{d} N(0, \Sigma)$, where $\Sigma = \sigma^2 \tilde{\Sigma}$ with $\tilde{\Sigma}_{kk} = \min\{c_0, 1\} + g_{c_0, K}$ and $\tilde{\Sigma}_{kl} = \min\{c_0, 1\}$ for $k \neq l$. Because $\tilde{\Sigma}$ does not depend on σ, and σ enters as a multiplicative factor, one can decorrelate $\hat{\tau}_1, \ldots, \hat{\tau}_K$ without estimating the LRV.[13]* □

3.2 Choosing K

The test statistic $\mathbb{T}_K$ and its limiting distribution depend on K. Choosing K seems unavoidable; it is inherent to the cross-fitting procedure required for bias correction. The choice of K is subject to a trade-off between the expected length of the confidence intervals and their finite sample coverage properties.

Figure 3 illustrates this trade-off. It shows that choosing a larger K will lead to shorter confidence intervals but may impact the coverage accuracy of our method. The reason for the loss in coverage accuracy is that we are constructing the bias correction term in (5) by averaging over a shorter time period, which lowers the quality of the normal approximation in Theorem 1 and leads to increased finite sample dependence between the blocks. See, for example, Ibragimov and Müller (2010, Section 2.3) for a related discussion in the context of the standard t-test with dependent data.

To formalize the trade-off between coverage accuracy and length, it is helpful to analyze the asymptotic efficiency of the confidence intervals in (9) by comparing the expected asymptotic length for a fixed K to the limiting case where $K \to \infty$. While our theory requires K

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[13]Canay et al. (2017) show that the sign-based randomization test outperforms the t-test when there is heterogeneity in the variances of the component estimators. However, in the presence of such heterogeneity, decorrelating $\hat{\tau}_1, \ldots, \hat{\tau}_K$ would require estimating the LRV for each component estimator, which is difficult in small samples.

Figure 3: Trade-off between coverage accuracy and length when choosing K

Coverage 90% confidence intervals

Average length 90% confidence intervals
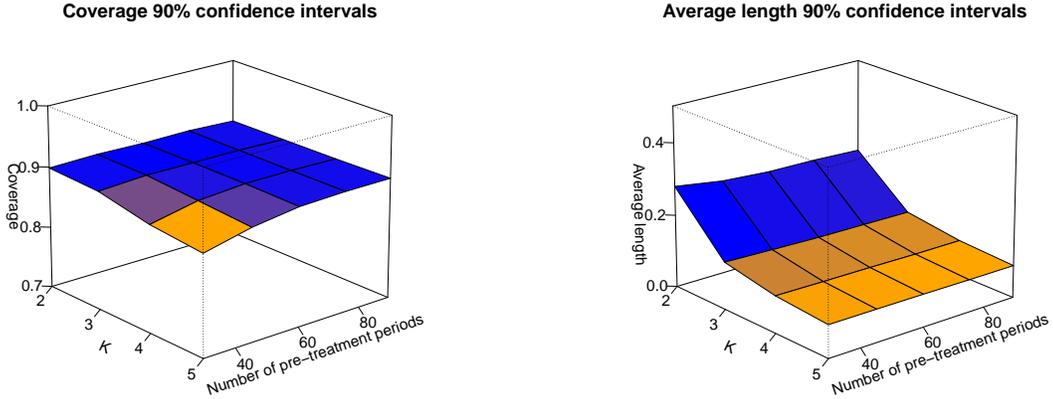
Notes: Simulations with 20,000 repetitions based on DGP1 described in Section 5. Because the trade-off is more relevant and pronounced when the persistence is higher, we show results for $\rho_u = 0.6$ for the purpose of illustration ($\rho_u = 0.31$ in Section 5). Nominal coverage: $1 - \alpha = 0.9$. We set $(T_1, N) = (16, 14)$ as in the empirical application and vary T_0 and K.

to be fixed, the case where $K \to \infty$ is a useful theoretical benchmark.

The length of the confidence interval (9) is $L = t_{K-1}(1 - \alpha/2)\hat{\sigma}_{\hat{\tau}}/\sqrt{K}$. In Appendix E, we show that $\sqrt{\min\{T_0, T_1\}}L \xrightarrow{d} \mathcal{L}$, with

$$E(\mathcal{L}) = 2\sigma t_{K-1}(1 - \alpha/2)\frac{1}{\sqrt{K}\sqrt{K-1}}\sqrt{1 + \min\{c_0, K\}}\sqrt{g_{c_0, K}}\sqrt{2}\frac{\Gamma(K/2)}{\Gamma((K-1)/2)}, \quad (12)$$

where $\Gamma(\cdot)$ denotes the Gamma function. Using Stirlings's approximation of the Gamma function, we obtain the following limiting length as $K \to \infty$,

$$E(\mathcal{L}) \approx 2\sigma\Phi^{-1}(1 - \alpha/2)\sqrt{\min\{c_0^{-1}, 1\}}\sqrt{1 + c_0}, \quad (13)$$

where $\Phi^{-1}(\cdot)$ is the quantile function of the standard normal distribution. See Appendix E for detailed derivations. The relative asymptotic efficiency (RAE) can then be computed as the ratio of (13) and (12),

$$\text{RAE} = \frac{\Phi^{-1}(1 - \alpha/2)\sqrt{\min\{c_0^{-1}, 1\}}\sqrt{1 + c_0}}{t_{K-1}(1 - \alpha/2)\frac{1}{\sqrt{K}\sqrt{K-1}}\sqrt{1 + \min\{c_0, K\}}\sqrt{g_{c_0, K}}\sqrt{2}\frac{\Gamma(K/2)}{\Gamma((K-1)/2)}}, \quad (14)$$

and is a function of α, c_0, and K only. Table 1 shows the RAE for $c_0 = T_0/T_1 = 30/16$, as in the empirical application in Section 6.

Table 1 shows that choosing $K = 3$ almost doubles the RAE relative to $K = 2$. However, this choice can be conservative: increasing K to 4, 5, or even 6 improves RAE by 12, 19, and 22 percentage points, respectively. The RAE gains from increasing K further are decreasing. Importantly from a practical perspective, one can achieve a very high RAE without choosing a large K: choosing $K = 9$ already yields confidence intervals with more than 90% RAE.

Table 1: Relative asymptotic efficiency

K	2	3	4	5	6	7	8	9	10
RAE	32.65	63.56	75.86	82.08	85.79	88.23	89.97	91.26	92.25

Notes: RAE: relative asymptotic efficiency as defined in (14). Results are shown for $\alpha = 0.1$ and $c_0 = 30/16$, as in the empirical application.

The RAE formula captures one side of the trade-off. The other side of the trade-off, coverage accuracy, can be assessed using application-based simulations. The simulations reported in this paper (based on Andersson (2019a)) and in an earlier version (Chernozhukov et al. (2022), based on Abadie and Gardeazabal (2003)) suggest that $K = 3$ works well when T_0 is small. When T_0 is moderate or large, the coverage accuracy remains excellent even for larger values of K, which justifies choosing larger values of K.

An important determinant of the coverage accuracy is the degree of persistence in the prediction errors $\{u_t\}$: the more persistent the prediction errors, the lower the coverage accuracy. A simple approach for gauging the persistence in $\{u_t\}$ is to fit an AR(1) model to the SC residuals in the pre-treatment period, as we do in Section 5. Assessing the persistence in this way is a helpful first step in evaluating coverage accuracy and an essential input for application-based simulations.

In summary, $K = 3$ is a useful starting point in typical SC applications where T_0 is small. If the estimated persistence in $\{u_t\}$ is relatively low and simulations indicate a good coverage accuracy, choosing $K = 4$ improves the efficiency of the t-test. If T_0 is moderate or large, K can often be chosen to achieve 80% RAE or even 90% RAE without affecting the coverage properties too much.

3.3 Debiased synthetic control is more efficient than difference-in-differences

Here we show that bias-corrected SC is more efficient than DID.[14] To illustrate, suppose that $T_0 < T_1$. The DID estimator of the ATT can be written as (e.g., Doudchenko and Imbens, 2016),

$$\hat{\tau}^{\text{DID}} = \frac{1}{T_1} \sum_{t=T_0+1}^{T} (Y_t - X_t' w_{\text{DID}}) - \frac{1}{T_0} \sum_{t=1}^{T_0} (Y_t - X_t' w_{\text{DID}}),$$

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[14]The recent work by Bottmer et al. (2021) complements our results by studying the efficiency of SC methods under design-based uncertainty.

where $w_{\text{DID}} := \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)'$. For simplicity, suppose that $E(Y_t(0)) = 0$ and $E(X_t) = 0$ and assume that the data are iid. Then a CLT gives

$$\sqrt{T_0}(\hat{\tau}^{\text{DID}} - \tau) \xrightarrow{d} N(0, (c_0 + 1)\sigma^2_{\text{DID}}), \quad \sigma^2_{\text{DID}} = E\left(Y_t(0) - X_t'w_{\text{DID}}\right)^2.$$

For SC, the previous results imply

$$\sqrt{T_0}(\hat{\tau} - \tau) \xrightarrow{d} N(0, (c_0 + 1)\sigma^2_{SC}), \quad \sigma^2_{SC} = E(Y_t(0) - X_t'w_*)^2,$$

where $w_* = \arg\min_{v \in \mathcal{W}^{SC}} E(Y_t(0) - X_t'w)^2$. To show that the SC estimator is more efficient than DID, it suffices to note that

$$\sigma^2_* = E(Y_t(0) - X_t'w_*)^2 = \min_{w \in \mathcal{W}^{SC}} E(Y_t(0) - X_t'w)^2 \le E\left(Y_t(0) - X_t'w_{\text{DID}}\right)^2 = \sigma^2_{\text{DID}}.$$

The above inequality shows that the magnitude of the efficiency gain is determined by $\sigma^2_{\text{DID}} - \sigma^2_*$, which depends on the true data-generating process. We note that the results in this section are asymptotic. In small samples, the uncertainty from estimating the weights can mask the asymptotic efficiency improvements, especially when these efficiency improvements are relatively small, i.e., when $\sigma^2_{\text{DID}} - \sigma^2_*$ is small.

4 Theoretical properties with non-stationary data

In many applications the data are non-stationary. An important rationale for applying SC is that the precise structure of the nonstationarity is unknown such that one has to rely on controls to identify counterfactual trends. Therefore, unlike the existing literature (e.g., Li, 2020), we do not make assumptions on the exact form of nonstationarity and instead restrict the heterogeneity in the nonstationarity across units. In essence, we require the units to be sufficiently homogeneous, which is an important contextual requirement in SC studies (Abadie, 2021).

4.1 Unrestricted common nonstationarity

We start by establishing the validity of our procedure under the following general class of nonstationary processes. As in Section 3, we allow for arbitrary misspecification.

Assumption 4. *For $1 \le t \le T$, $Y_t(0) = V_t(0) + \theta_t$ and $X_t = Z_t + \mathbf{1}_N \theta_t$, where $\theta_t \in \mathbb{R}$ is an unrestricted stochastic process and $\{(V_t(0), Z_t)\}_{t=1}^T$ is covariance-stationary.*

Assumption 4 requires that the potential outcomes for the treated unit and all control units can be decomposed into a common non-stationary component θ_t and a stationary

component. Importantly, $\{\theta_t\}$ can be an arbitrary stochastic process (e.g., a deterministic trend, random walk, or general ARIMA process) such that researchers are not required to impose further restrictions on the nonstationarity as long as it is shared among units. This feature adds very useful robustness. Even for low-dimensional problems it is well-known that the slightest misspecification in modeling nonstationarity (e.g., unit root vs. near unit root) can yield invalid inferences.[15]

To derive the asymptotic properties of our method under Assumption 4, we exploit the specific structure of the SC estimator and the constraints on the weights. First, the estimated and the pseudo-true SC weights satisfy $\mathbf{1}'_N \hat{w}_{(k)} = \mathbf{1}'_N w_* = 1$. Therefore, under Assumption 4, we have a version of Lemma 1 with $\tilde{X}_t$ replaced by $\tilde{Z}_t$, which is stationary.

Second, $\hat{w}_{(k)}$ is ℓ_2-consistent under Assumption 4. Since $\mathcal{W}^{SC} \subset \{w : \mathbf{1}'_N w = 1\}$, the estimated and the pseudo-true SC weights can be expressed in terms of the stationary parts of the potential outcomes, $(V_t(0), Z_t)$:

$$\hat{w}_{(k)} = \arg\min_{w \in \mathcal{W}^{SC}} \sum_{t \in H_{(-k)}} (Y_t(0) - X_t' w)^2 = \arg\min_{w \in \mathcal{W}^{SC}} \sum_{t \in H_{(-k)}} (V_t(0) - Z_t' w)^2 \qquad (15)$$

$$w_* := \arg\min_{w \in \mathcal{W}^{SC}} E(Y_t(0) - X_t' w)^2 = \arg\min_{w \in \mathcal{W}^{SC}} E(V_t(0) - Z_t' w)^2 \qquad (16)$$

Therefore, if the conditions in Lemma 2 hold with $(Y_t(0), X_t)$ replaced by $(V_t(0), Z_t)$, consistency of $\hat{w}_{(k)}$ follows. The following lemma states the formal result.

Lemma 3. *Suppose that the assumptions of Lemma 2 hold with $(Y_t(0), X_t)$ replaced by $(V_t(0), Z_t)$. Then Assumption 2 holds.*

Finally, by (15), $\hat{w}_{(k)}$ is only a function of $\{Z_t\}_{t \in H_{(-k)}}$ under Assumption 4. Thus, under weak dependence of $\{Z_t\}$, $\hat{w}_{(k)} - w$ is approximately independent of $\{Z_t\}_{t \in H_k \cup \{T_0+1,\ldots,T\}}$, so that the asymptotic properties of our method can be established using similar arguments as in the proofs of Theorems 1 and 2.

Theorem 3. *Let Assumptions 2, 3 with X_t replaced by Z_t, and 4 hold. Suppose that $T_0/T_1 \to c_0$ for some $c_0 \in [0, \infty]$. Then (i) $\mathbb{T}_K \xrightarrow{d} t_{K-1}$ and (ii) $P(\tau \in \mathcal{I}_K(1-\alpha)) \to 1 - \alpha$.*

Theorem 3 formally guarantees the validity of the confidence interval $\mathcal{I}_K(1-\alpha)$ when the data exhibit a common unrestricted nonstationarity.

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[15]For example, there is a large literature on making inference on the scalar parameter β in $Y_t = X_t\beta + U_t$, where $X_t = \rho X_{t-1} + Z_t$ and $\rho = 1 + c/T$. The key difficulty is that the asymptotic distribution of the usual t-statistic for β depends on c, but c cannot be consistently estimated (e.g., Phillips, 2014). Hence, using a unit root process ($c = 0$) for the dynamics of x_t is not robust to barely undetectable misspecification (e.g., local-to-unity $c < 0$).

4.2 Deviations from common nonstationarity

Here we study a more general setting that allows for deviations from the common nonstationarity. Unlike in the previous sections, we do not allow SC to be misspecified. Consider the following assumption.

Assumption 5. *Assume that $X_t = \mathbf{1}_N \theta_t + Z_t + \beta \xi_t$, $1 \leq t \leq T$, where θ_t is an unrestricted stochastic process, $E(Z_t) = 0$, ξ_t is a non-stationary process, $\beta \in \mathbb{R}^N$ is an arbitrary vector. Assume that $\{\theta_t\}_{t=1}^T$, $\{Z_t\}_{t=1}^T$, and $\{\xi_t\}_{t=1}^T$ are mutually independent. Moreover, $Y_t(0) = X_t'w + u_t$, $1 \leq t \leq T$, where $E(u_t) = 0$, $w \in \mathcal{W}^{SC}$.*

Assumption 5 considers deviations from Assumption 4. The deviations are driven by the non-stationary process ξ_t. Assumption 5 does not impose any restrictions on the vector β. This allows us to accommodate many leading examples of non-stationary data. For example, the control units can have different deterministic trends. Moreover, Assumption 5 allows for any cointegration system whose nonstationarity is driven by θ_t and ξ_t. The case with multiple ξ_t's would be much more complicated since the analysis would need to take into account the correlation among them and whether they diverge at the same rate; we leave this extension for future research.

The following assumption restricts the relative magnitude of T_0, T_1, and N, as well as the magnitude of the deviation process ξ_t.

Assumption 6. *Assume $T_1 \ll T_0/\log N$. Suppose that*

$$\frac{\sum_{t=T_0+1}^T \xi_t}{\sqrt{\sum_{t \in H_{(-k)}} \xi_t^2}} = O_P(T_1 T^{-1/2}) \qquad and \qquad \frac{\sum_{t \in H_k} \xi_t}{\sqrt{\sum_{t \in H_{(-k)}} \xi_t^2}} = O_P(T_1 T^{-1/2}).$$

With a common nonstationarity, no restrictions on the relative magnitude of T_0 and T_1 are required (Section 4.1). However, to accommodate deviations form a common nonstationarity, we require T_0 to be much larger than T_1. The restriction on the magnitude of ξ_t can be verified for many non-stationary processes including unit roots and polynomial trends; see Appendix F for details.

Assumptions 5 and 6 allow for general non-stationary trends and deviations without imposing specific structures on the dynamics. This is important in practice because procedures that rely on specific assumptions on the nonstationarity are typically not robust against deviations from these assumptions, and even the slightest misspecification can yield invalid inferences, as discussed above.

Finally, we impose conditions on the stationary component Z_t.

Assumption 7. *Let Assumption 3 hold with X_t replaced by Z_t. In addition, suppose that the following conditions hold.*

1. $\|\sum_{t \in H_{(-k)}} Z_t u_t\|_\infty = O_P(\sqrt{T_0 \log N})$

2. $\|\sum_{t \in H_{(-k)}} Z_t \xi_t\|_\infty / \sqrt{\sum_{t \in H_{(-k)}} \xi_t^2} = O_P(\sqrt{\log N})$

3. $(\sum_{t \in H_{(-k)}} u_t \xi_t) / \sqrt{\sum_{t \in H_{(-k)}} \xi_t^2} = O_P(1)$

4. $\|\sum_{t \in H_{(-k)}} (Z_t Z_t' - E Z_t Z_t')\|_\infty = O_P(\sqrt{T_0 \log N})$

The next theorem establishes the asymptotic validity of the proposed inference method when there are sparse deviations.

Theorem 4. *Let Assumptions 5, 6, and 7 hold and suppose that we use the SC estimator (6). Then, for $1 \le k \le K$,*

$$\sqrt{T_1}(\hat{\tau}_k - \tau) = \frac{1}{\sqrt{T_1}} \sum_{t=T_0+1}^{T} u_t - \frac{1}{\sqrt{|H_k|}} \sum_{t \in H_k} u_t + o_P(1).$$

Moreover, $\mathbb{T}_K \xrightarrow{d} t_{K-1}$ and $P(\tau \in \mathcal{I}_K(1-\alpha)) \to 1 - \alpha$.

To our knowledge, Theorem 4 guarantees the inference validity of SC under the most general available conditions without imposing a specific structure on the dynamics of the nonstationarity.

Remark 7. *DID is generally inconsistent under deviations from a common nonstationarity. For example, suppose that there is a linear trend ($\xi_t = t$). The DID estimator would have a bias of $T_1(T_0 + (T_1+1)/2)(w'\beta - N^{-1}\mathbf{1}_N'\beta)$. Thus, DID is inconsistent unless $w'\beta - N^{-1}\mathbf{1}_N'\beta = o(T^{-1}T_1^{-1})$.* $\square$

5 Application-based simulations

It is important to understand the advantages and limitations of the t-test in practice. Here we present simulation evidence demonstrating when it works well and when it does not. We calibrate the simulations to the empirical application in Section 6, where we revisit the analysis of the effect of carbon taxes on emissions in Andersson (2019a). We show results for $(T_0, T_1, N) = (30, 16, 14)$ as in Andersson (2019a). In Appendix H, we further investigate the performance of the t-test with $T_0 = 150$, because the theoretical results in Section 4.2 require T_0 to be much larger than T_1, and the performance with $T_1 \in \{10, 12, 14, 16\}$. All simulations were carried out in R (R Core Team, 2023).

We generate the treated outcome as

$$Y_t(0) = \mu + X_t'w + u_t, \ u_t = \rho_u u_{t-1} + v_t, \ v_t \overset{iid}{\sim} N(0, \sigma_v^2), \ 1 \le t \le T,$$

and set $\alpha_t = \tau = 0$. The parameters (μ, w) vary across DGPs, and σ_v^2 and ρ_u are obtained by fitting an AR(1) model to the empirical SC residuals. The degree of persistence in $\{u_t\}$ is relatively low: $\rho_u = 0.31$. To generate the control outcomes, we fit a factor model with four factors to the detrended data and let $Y_{it}(0) = \theta_{it} + L_i'F_t + \eta_{it}$, where θ_{it} is a unit-specific non-stationary component that varies across DGPs, $F_t = (F_{1t}, \ldots, F_{4t})'$, $F_{st} \overset{iid}{\sim} N(0, \sigma_{F_s}^2)$, $\eta_{it} = \rho_i \eta_{it-1} + \epsilon_{it}$, and $\epsilon_{it} \sim N(0, \sigma_{\epsilon_i}^2)$. $(L_1, \ldots, L_N)$, $(\sigma_{F_1}^2, \ldots, \sigma_{F_4}^2)$, $(\rho_1, \ldots, \rho_N)$, and $(\sigma_{\epsilon_1}^2, \ldots, \sigma_{\epsilon_N}^2)$ are obtained from and estimated based on the factor model fitted to the data.

We consider three stationary and six non-stationary DGPs; see Table 2. DGP1–DGP3 satisfy the assumptions in Section 3. DGP4–DGP5 satisfy the conditions in Section 4.1 since $\mathbf{1}_N' w^{\text{MIS}} = 1$, and DGP6–DGP7 satisfy the assumptions in Section 4.2. DGP8 and DGP9 allow for studying the performance of the t-test in settings not covered by our theory. While the nonstationarity in DGP8 satisfies Assumption 5, which allows for unit-specific trends, our theory does not allow for misspecification when there are deviations from a common nonstationarity. DGP9 captures a setting where the deviations are driven by multiple non-stationary processes, whereas Assumption 5 requires $\{\xi_t\}$ to be a scalar-valued process.

Table 2: DGPs

	Stationary DGPs	
DGP1	$\mu = 0$, $w = w^{\text{SC}}$, $\theta_{it} = 0$	SC weights
DGP2	$\mu = 0$, $w = w^{\text{DID}}$, $\theta_{it} = 0$	DID weights
DGP3	$\mu = 2$, $w = w^{\text{MIS}}$, $\theta_{it} = 0$	Misspecified
	Non-stationary DGPs	
DGP4	$\mu = 2$, $w = w^{\text{MIS}}$, $\theta_{it} = t$	Common linear trend & misspecification
DGP5	$\mu = 2$, $w = w^{\text{MIS}}$, $\theta_{it} = \theta_t$, $\theta_t = \theta_{t-1} + \xi_t, \xi_t \overset{iid}{\sim} N(0,1)$	Common random walk & misspecification
DGP6	$\mu = 0$, $w = w^{SC}$, $\theta_{it} = t + \mathbf{1}\{i = 1\} \cdot t$	Common trend & deviation
DGP7	$\mu = 0$, $w = w^{SC}$, $\theta_{it} = \theta_{1t} + \mathbf{1}\{i = 1\} \cdot \theta_{2t}$,	Common random walk & deviation
	$\theta_{1t} = \theta_{1t-1} + \xi_{1t}, \xi_{1t} \overset{iid}{\sim} N(0,1)$,	
	$\theta_{2t} = \theta_{2t-1} + \xi_{2t}, \xi_{2t} \overset{iid}{\sim} N(0,1)$	
DGP8	$\mu = 0$, $w = w^{\text{MIS}}$, $\theta_{it} = i + i \cdot t$	Heterogeneous trends & misspecification
DGP9	$\mu = 0$, $w = w^{\text{SC}}$, $\theta_{it} = i + \theta_{it-1} + \xi_{it}, \xi_{it} \overset{iid}{\sim} N(0,1)$	Random walks with heterogeneous drifts

Notes: w^{SC}: SC estimate of w. $w^{\text{DID}} = (1/N, \ldots, 1/N)'$, $w^{\text{MIS}} = (-3, 3, 1, \ldots, 0)'$ (such that $\mathbf{1}_N' w^{\text{MIS}} = 1$, which ensures that DGP4–DGP5 satisfy the assumptions in Section 4.1).

We compare the t-test to four alternative methods for making inferences on the ATT:

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(i) DID with K-fold cross-fitting, (ii) subsampling inference based on $\hat{\tau}^{SC}$ (Li, 2020),[16] (iii) SDID (Arkhangelsky et al., 2021) implemented using the R-package synthdid (Hirshberg, 2023), and (iv) K-fold cross-fitting based on the true w (Oracle). In Appendix H.3, we also compare the t-test to the widely-used permutation approach by Abadie et al. (2010).

Table 3 shows the bias, coverage, and average length for all methods and DGPs. The nominal coverage is $1 - \alpha = 0.9$. Under stationarity, our t-test exhibits an excellent performance, even when $T_0 = 30$. Under correct specification, bias, coverage, and average length of the confidence intervals are similar to the Oracle. Consistent with our theory, the t-test effectively removes the bias under misspecification. Misspecification does not affect coverage accuracy but leads to wider confidence intervals because it increases the variance of the prediction errors.

When there is a common nonstationarity, the t-test performs well and is fully robust to misspecification, consistent with our theory. When there are deviations from the common nonstationarity and $T_0 = 30$, it may exhibit some undercoverage, especially if there is also misspecification (for example, under DGP8, which is not covered by our theory). This is expected since our theoretical results under deviations from nonstationarity require T_0 to be much larger than T_1. As shown in the Appendix H.1, when $T_0 = 150$, the t-test works well under all DGPs covered by our theory. The results for DGP8 and DGP9 even suggest that if T_0 is large enough, the t-test remains quite robust in settings not covered by our theory.

The t-test performs well compared to the alternative methods. It is more robust than DID, which exhibits large biases under deviations from a common nonstationarity, and can yield substantially shorter confidence intervals than DID (DGP6–DGP9).[17] Subsampling undercovers even under correct specification when the bias is negligible. When SC is misspecified, $\hat{\tau}^{SC}$ is typically biased (see also Figure 1), which can result in zero coverage. The t-test demonstrates a better coverage accuracy overall, and removes the bias due to misspecification. Finally, while SDID is nearly as effective at removing the bias as cross-fitting, it exhibits undercoverage due to the confidence intervals being too short or overcoverage due to the confidence intervals being too long under most DGPs. When interpreting the results of this simulation comparison, it is important to note that SDID with one treated unit relies

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[16]Our R-implementation is based on Matlab code obtained from Kathleen Li. We take the subsample size to be $(2/3) \cdot T_0$, corresponding to the middle choice of $m = 60$ in Section 5 of Li (2020).

[17]We note that our application-based DGPs are quite favorable to DID: even when the data are stationary and SC is correctly specified, the DID confidence intervals are not much wider than those from the t-test. This finding suggests that the asymptotic efficiency gains from using the t-test are limited under these specific DGPs. By contrast, in simulations based on Abadie and Gardeazabal (2003), we found that the DID confidence intervals can be much wider than those from the t-test, even when DID is theoretically valid (Chernozhukov et al., 2022, Section 5).

on cross-sectional variation and homoskedasticity for inference, whereas the t-test relies on time series variation. These two sets of assumptions are non-nested. See Sections 1.1 and 7.1 for further discussions.

Table 3: Simulation results with $(T_0, T_1, N) = (30, 16, 14)$

	Bias×10					Coverage					Average length CI				
K	SC	DID	SuSa	SDID	Oracle	SC	DID	SuSa	SDID	Oracle	SC	DID	SuSa	SDID	Oracle
	DGP1 (stationary, SC weights)														
3	0.00	0.01	-0.00	-0.01	0.00	0.91	0.90	0.77	0.92	0.89	0.09	0.11	0.03	0.10	0.07
4	-0.00	-0.01	-0.00	-0.01	-0.00	0.90	0.89	0.77	0.92	0.89	0.07	0.09	0.03	0.10	0.06
	DGP2 (stationary, DID weights)														
3	-0.00	-0.00	-0.00	-0.01	-0.00	0.90	0.89	0.76	0.93	0.89	0.09	0.07	0.04	0.10	0.07
4	-0.00	-0.00	-0.00	-0.01	-0.00	0.91	0.89	0.76	0.93	0.89	0.07	0.06	0.04	0.10	0.06
	DGP3 (stationary, misspecified)														
3	-0.00	-0.02	19.99	-0.05	0.00	0.90	0.90	0.00	0.69	0.89	0.68	0.63	0.32	0.10	0.07
4	0.02	0.02	19.99	-0.05	0.00	0.90	0.90	0.00	0.69	0.89	0.56	0.53	0.32	0.10	0.06
	DGP4 (common linear trend, misspecified)														
3	-0.00	-0.01	20.01	-0.01	-0.00	0.89	0.90	0.00	0.71	0.90	0.67	0.63	0.32	0.10	0.08
4	0.02	-0.00	20.01	-0.01	-0.00	0.89	0.89	0.00	0.71	0.90	0.56	0.53	0.32	0.10	0.06
	DGP5 (common random walk, misspecified)														
3	0.02	0.01	19.98	0.02	0.00	0.90	0.90	0.00	0.72	0.89	0.68	0.63	0.32	0.10	0.07
4	0.04	0.02	19.98	0.02	-0.00	0.89	0.89	0.00	0.72	0.89	0.56	0.53	0.32	0.10	0.06
	DGP6 (common linear trend & deviation, SC weights)														
3	-0.07	-16.43	-0.09	-0.03	-0.00	0.81	1.00	0.74	1.00	0.89	0.08	4.08	0.04	18.83	0.07
4	-0.05	-15.71	-0.09	-0.03	0.00	0.78	0.00	0.74	1.00	0.88	0.06	2.52	0.04	18.83	0.06
	DGP7 (common random walk & deviation, SC weights)														
3	-0.00	0.05	0.00	0.04	-0.00	0.88	0.77	0.75	0.98	0.90	0.10	0.79	0.04	2.61	0.07
4	-0.00	-0.01	0.00	0.04	0.00	0.86	0.64	0.75	0.98	0.89	0.07	0.54	0.04	2.61	0.06
	DGP8 (heterogeneous trends, misspecified)														
3	-0.02	-345.03	-0.00	-0.74	0.00	0.59	1.00	0.80	1.00	0.89	0.33	85.77	0.42	10.59	0.07
4	0.04	-329.97	-0.00	-0.74	-0.00	0.64	0.00	0.80	1.00	0.89	0.35	52.90	0.42	10.59	0.06
	DGP9 (random walks with heterogeneous drifts, SC weights)														
3	-0.02	-369.70	0.00	0.01	-0.00	0.92	1.00	0.81	1.00	0.90	0.53	91.81	0.21	15.05	0.08
4	0.01	-353.62	0.00	0.01	0.00	0.82	0.00	0.81	1.00	0.90	0.26	56.76	0.21	15.05	0.06

Notes: Simulation design based on the empirical application as described in the main text. Simulations are based on 5,000 repetitions, except for SDID for which we use 200 repetitions due to the placebo procedure being computationally expensive. CI: Confidence interval. Nominal coverage: $1 - \alpha = 0.9$.

6   Estimating the impact of carbon taxes on emissions

In this section we revisit the SC analysis of the causal effect of carbon taxes on CO2 emissions in Andersson (2019a). Andersson exploits the introduction of a carbon tax on transport fuels

during the early 1990s in Sweden, using $J = 14$ OECD countries as control units.[18] The outcome variable of interest measures CO2 emissions from transport (in metric tons per capita). The data are annual panel data from 1960–2005.[19] Andersson uses data up to 2005 because the EU emissions trading system started in that year. The pre-treatment period is 1960–1989, and the post-treatment period is 1990–2005, so that $(T_0, T_1) = (30, 16)$.

Figure 4 displays the raw data. It shows a drop right before 1990 and reduced growth in emissions afterward. We apply the t-test to investigate whether this drop and reduced growth are due to the carbon tax and compare the results to other methods. All computations were performed in R (R Core Team, 2023).

Figure 4: Raw data on emissions per capita (Andersson, 2019a,b)
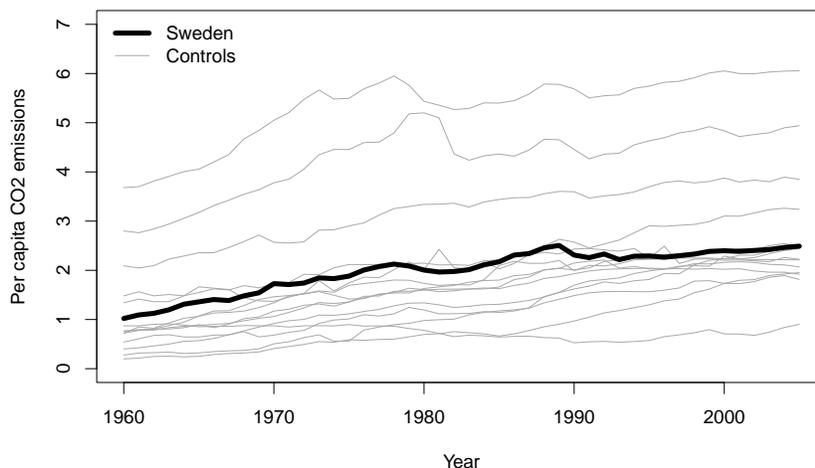
We start by assessing the validity of the t-test using the placebo test, which is discussed in more detail in Appendix B.1. We split the pre-treatment period into a placebo pre-treatment period, $\{1, \ldots, \widetilde{T}_0\}$, and a placebo post-treatment period, $\{\widetilde{T}_0 + 1, \ldots, T_0\}$. Under the null hypothesis that the assumptions underlying our method are valid, the confidence intervals should contain zero. Table 4 shows the placebo ATT estimates and the confidence intervals for $K = 3$ and $\widetilde{T}_0 \in \{18, 21\}$ (to ensure a long-enough time period for estimating the weights and divisibility by $K = 3$). The placebo estimates are smaller in magnitude than the actual effect estimates in Table 5 below, and the confidence intervals include zero. Thus, we do not reject the validity of the assumptions underlying the t-test.

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[18]The countries are: "Australia, Belgium, Canada, Denmark, France, Greece, Iceland, Japan, New Zealand, Poland, Portugal, Spain, Switzerland, and the United States" (Andersson, 2019a, p.9).

[19]The data are available in the replication package (Andersson, 2019b).

Table 4: Results placebo test

$\widetilde{T}_0$	ATT	90%-CI	
18	0.01	-0.18	0.19
21	0.10	-0.21	0.41

Notes: Table shows point estimates and 90% confidence intervals for the placebo test described in Appendix B.1.

Table 5 shows the point estimates of the ATT and 90% confidence intervals. As discussed in Section 3.2, $K = 3$ constitutes a useful benchmark in settings where T_0 is small or moderate. The simulations in Section 5 further show that the t-test performs well even when $K = 4$. We therefore report results for $K \in \{3, 4\}$. The point estimates are equivalent (up to rounding) for both values of K. The confidence interval for $K = 3$ is somewhat wider than the confidence interval for $K = 4$, and neither interval includes zero. Thus, our findings corroborate the results in Andersson (2019a) based on the permutation test of Abadie et al. (2010).[20]

Table 5: Results t-test

K	ATT	90%-CI	
3	-0.27	-0.41	-0.14
4	-0.27	-0.36	-0.19

Notes: Point estimates and 90% confidence intervals based on the t-test with $K \in \{3, 4\}$.

Table 6 compares the results of the t-test with $K = 3$ to DID with $K = 3$, subsampling, and SDID. The point estimates are similar across all methods, ranging from -0.35 for SDID to -0.21 for DID. Consistent with the t-test, the results based on DID and subsampling suggest that the carbon tax significantly decreased emissions. SDID yields wider confidence intervals that include zero. This finding is in line with the simulations in Section 5, which show that SDID may yield wider confidence intervals than the t-test when there is heterogeneity in the nonstationarities across control units.

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[20]Note that our SC specification differs from Andersson (2019a). We use all past outcomes as predictors, whereas Andersson (2019a) uses a subset of past outcomes and additional predictors to compute the weights.

Table 6: Comparison to DID, subsampling, and SDID

SC ($K = 3$)			DiD ($K = 3$)			Subsampling			Synthetic DiD		
ATT	90%-CI		ATT	90%-CI		ATT	90%-CI		ATT	90%-CI	
-0.27	-0.41	-0.14	-0.21	-0.36	-0.07	-0.28	-0.37	-0.23	-0.35	-0.84	0.14

Notes: Point estimates and 90% confidence intervals for different methods. DID, subsampling, and SDID are implemented as described in Section 5. Note that the subsampling CI is not symmetric due to being based on the percentile method.

7   Recommendations for practice

7.1   When should you use the t-test?

Before deciding whether to use the proposed t-test, researchers need to decide whether to use SC in the first place. Abadie (2021) provides a detailed discussion of the relevant practical considerations. The most popular alternative to SC is DID. While conventional SC and DID are non-nested (Doudchenko and Imbens, 2016), we show that bias-corrected SC is more robust than DID. Therefore, we recommend it whenever the DID assumptions are questionable.

There are three important and interrelated considerations when choosing a suitable method for making inferences on the ATT using the SC method. See Section 1.1 for references to papers proposing inference methods for per-period effects and sharp null hypotheses.

Number of treated units.   The t-test is designed for applications with one treated unit. When there are multiple treated units, the t-test can be applied separately for each unit. However, when the number of treated units is large, other approaches that directly target average effects across units, such as Abadie and L'Hour (2021) and Arkhangelsky et al. (2021), might be preferable. For applications with multiple treated units and staggered treatment adoption, we recommend using methods that are specifically designed to accommodate staggered adoption, such as Shaikh and Toulis (2021), Ben-Michael et al. (2022), or Cattaneo et al. (2023).

Number of periods.   Existing asymptotic inference methods that rely on classical estimators of the LRV or subsampling can exhibit substantial size distortions in small samples. By contrast, the t-test, which relies on a self-normalized test statistic, exhibits higher-order improvements and performs well in simulations when T_0 and T_1 are small. Therefore, we

recommend using the t-test instead of methods that rely on estimating the LRV or subsampling when T_0 and T_1 are small or moderate, as is the case in many SC applications. When T_1 is too small to rely on asymptotics where $T_1 \to \infty$, researchers can use inference procedures that are valid when T_1 is fixed, such as Chernozhukov et al. (2021) or Cattaneo et al. (2023).[21]

Which type of variation to exploit for inference. In panel data settings, inference methods can exploit the time series and/or the cross-sectional dimension. The t-test exploits the time series dimension for inference. It relies on stationarity and weak dependence of the SC prediction errors for the treated unit over time and is therefore particularly well-suited for settings where the units are heterogeneous. This is often the case in SC applications based on aggregate units, such as states or countries. For example, in our empirical application, the analysis is at the country level and the control units used to construct the counterfactual for Sweden are other OECD countries, including Iceland, Spain, and the United States.

The stationarity and weak dependence assumptions underlying the t-test are not innocuous and might be questionable when there are structural breaks in the data. If the units are homogeneous enough to justify homoskedasticity and weak dependence across units, we recommend methods that exploit cross-sectional variation, such as Arkhangelsky et al. (2021).

7.2 Recommendations for implementing the t-test

In Section 7.1, we discuss when to use the t-test. Here we provide some recommendations for implementing the t-test in empirical applications.

First, with non-stationary data, the robustness of the t-test improves substantially as T_0 increases. Therefore, we recommend collecting enough pre-treatment data in such settings. When collecting additional pre-treatment data, researchers need to be careful about structural breaks in the data. The placebo test described in Appendix B.1 can be used to test for structural breaks.

Second, the t-test relies on the predictive relationship between the treated and the control units being sufficiently stable over time. In Appendix B.1, we demonstrate that the t-test remains valid when the SC weights are varying over time in a stationary manner. For applications where researchers are concerned about the weights changing in non-stationary patterns, we recommend using the placebo test described in Appendix B.1 to assess the

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[21]See, for example, Masini and Medeiros (2020) and Masini and Medeiros (2021) for inference methods based on (penalized) regression approaches for estimating counterfactuals.

validity of the assumptions underlying the t-test. This test requires enough pre-treatment periods.

Finally, the choice of K is subject to an inherent trade-off between the expected length of the confidence intervals and their coverage accuracy. For SC applications with small and moderate sample sizes, choosing $K = 3$ provides a useful and robust benchmark: it ensures good coverage properties while providing a reasonable RAE. More generally, researchers can use the RAE formula (14) in conjunction with estimates of the degree of persistence in the prediction errors and application-based simulations to guide the choice of K. Section 3.2 provides a detailed discussion on how to choose K.

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Appendix (for online publication)

A Random treatment status

In the main text, we considered a setup where the treatment status is fixed. Here we discuss the properties of our method when the treatment status is random. In SC applications, the treatment assignment is usually not completely exogenous, and unconfoundedness conditions have been considered in the literature. The idea is that conditional on certain variables, the treatment assignment is independent (or mean-independent) of the potential outcomes. In Kellogg et al. (2021), the conditioning variables are allowed to include all the observed variables before the treatment; in works such as Athey et al. (2021), Ferman (2021), and Imbens and Viviano (2023) among others, the conditioning variables are latent variables.[22] Here we discuss our results under these two approaches. We treat the treatment status as random and the treatment date as fixed. Let $D \in \{0, \ldots, N\}$ denote a random variable indicating the identity of the treated unit. For $1 \leq t \leq T$ and $i \in \{0, \ldots, N\}$, we observe $Y_{it} = Y_{it}(1) \cdot \mathbf{1}\{D = i\} \cdot \mathbf{1}\{t \geq T_0 + 1\} + Y_{it}(0) \cdot (1 - \mathbf{1}\{D = i\} \cdot \mathbf{1}\{t \geq T_0 + 1\})$, where $Y_{it}(1) = Y_{it}(0) + \alpha_{it}$.

A.1 Selection on observed variables

We follow Kellogg et al. (2021) and allow the treatment assignment D to depend on observed variables, including the outcome variables in the pre-treatment period. Their paper establishes a decomposition of the bias of SC methods. We deviate from their paper in two ways. First, we allow the distribution of each unit to be different. Second, we impose additional assumptions to derive inference results for the ATT.

Assumption 8. *For $i \in \{0, \ldots, N\}$, $Y_{it}(0) = \theta_t + V_{it}$, where $\{V_t\}_{t=1}^{T}$ and $\{\theta_t\}_{t=1}^{T}$ are two independent processes with $V_t = (V_{0t}, \ldots, V_{Nt})' \in \mathbb{R}^{N+1}$. Moreover, $\{V_t\}_{t=1}^{T}$ is covariance-stationary and satisfies the following conditions:*

1. *There exists a constant $\kappa_1 > 0$ such that all the eigenvalues of $E V_t V_t'$ are in $[1/\kappa_1, \kappa_1]$*

2. *There exists a sequence $\rho_T > 0$ such that $P(\max_{1 \leq t \leq T} \|V_t\|_\infty \leq \rho_T) \to 1$*

3. *$\{V_t\}_{t=1}^{T}$ is β-mixing with coefficient satisfying $\beta_{\mathrm{mix}}(\gamma_T) \to 0$ for some γ_T with $0 < \gamma_T < r/2$ and $\rho_T \gamma_T = o(\min\{\sqrt{T_0}, \sqrt{T_1}\})$*

4. *$\max_{1 \leq k \leq K} \||H_{(-k)}|^{-1} \sum_{t \in H_{(-k)}} V_t V_t' - E V_t V_t'\|_\infty = o_P(1)$*

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[22]See Arkhangelsky and Hirshberg (2023) for some interesting recent work on combining both types of assumptions in SC settings with many treated units.

Assumption 8 allows for a common non-stationary trend θ_t and is similar to the conditions in Section 4.1. Our object of interest is

$$\tau = T_1^{-1} \sum_{i=0}^{N} \sum_{t=T_0+1}^{T} \alpha_{it} \cdot \mathbf{1}\{D = i\}. \tag{17}$$

Notice that only one unit is treated, and $D \in \{0, \ldots, N\}$ is allowed to be a random variable. Hence, this target is also random and our goal is to construct a prediction interval for τ. We impose the following unconfoundedness condition.

Assumption 9. *Let* $Y_t(0) = (Y_{0t}(0), \ldots, Y_{Nt}(0))' \in \mathbb{R}^{N+1}$. *Conditional on* $\{Y_t(0)\}_{T_0-B_T \leq t \leq T_0}$, D *is independent of* $\{Y_t(0)\}_{1 \leq t \leq T}$, *where* $B_T \to \infty$ *satisfies* $B_T \rho_T = o(\min\{\sqrt{T_0}, \sqrt{T_1}\})$.

Under Assumption 9, the selection into treatment can only depend on the outcome variables on B_T periods before the treatment date but does not depend on "ancient history". This might be reasonable if the treatment is a result of contemporary political, social and/or economic developments, such as public sentiment on tobacco or changes in geopolitical status. One can include additional pre-determined variables to the conditioning variables, and the same analysis applies once we write the assumptions conditional on these extra variables.

We define our procedure as follows. For $i \in \{0, \ldots, N\}$ and $k \in \{1, \ldots, K\}$, we define

$$\hat{w}_{(k)}^{(i)} \in \underset{w \in \mathcal{W}^{SC}}{\arg\min} \sum_{t \in H_{(-k)}} (Y_{it} - Y_{-i,t}' w)^2,$$

where $Y_{-i,t} \in \mathbb{R}^N$ is the vector $Y_t \in \mathbb{R}^{N+1}$ with the i-th component removed. For $k \in \{1, \ldots, K\}$, we define the estimator

$$\hat{\tau}_k = T_1^{-1} \sum_{i=0}^{N} \sum_{t=T_0+1}^{T} (Y_{it} - Y_{-i,t}' \hat{w}_{(k)}^{(i)}) \mathbf{1}\{D_i = 1\}.$$

In practice, we only need to compute the SC weights for the treated unit. Hence, the same construction as in Section 2.2 applies:

$$\mathbb{T}_K = \frac{\sqrt{K}(\hat{\tau} - \tau)}{\hat{\sigma}_{\hat{\tau}}},$$

where $\hat{\tau} = K^{-1} \sum_{k=1}^{K} \hat{\tau}_k$ and $\hat{\sigma}_{\hat{\tau}} = \sqrt{1 + Kr/T_1} \sqrt{(K-1)^{-1} \sum_{k=1}^{K} (\hat{\tau}_k - \hat{\tau})^2}$. The $(1 - \alpha)$ prediction interval for τ also takes the same form:

$$\mathcal{I}_K(1 - \alpha) = \left[\hat{\tau} - t_{K-1}(1 - \alpha/2) \frac{\hat{\sigma}_{\hat{\tau}}}{\sqrt{K}}, \ \hat{\tau} + t_{K-1}(1 - \alpha/2) \frac{\hat{\sigma}_{\hat{\tau}}}{\sqrt{K}} \right].$$

The following result establishes the validity of the procedure.

Theorem 5. *Let Assumptions 8 and 9 hold. Then* $\mathbb{T}_K \xrightarrow{d} t_{K-1}$ *and* $P(\tau \in \mathcal{I}_K(1 - \alpha)) \to 1 - \alpha$.

3

A.2 Selection on latent variables

Following the literature, we consider a factor model for the potential outcomes (e.g., Abadie et al., 2010; Arkhangelsky et al., 2021; Ben-Michael et al., 2021; Ferman, 2021; Ferman and Pinto, 2021),

$$Y_{it}(0) = \theta_t + L_i' F_t + \varepsilon_{it}, \quad E(\varepsilon_{it}) = 0, \quad 0 \le i \le N, \ 1 \le t \le T, \qquad (18)$$

where θ_t is a common unrestricted nonstationarity, L_i is a vector of loadings, F_t is a vector of factors, and ε_{it} is an idiosyncratic shock. Define $L := (L_0, \ldots, L_N)$, $F := (F_1, \ldots, F_T)$, and $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_T)$, where $\varepsilon_t = (\varepsilon_{0t}, \ldots, \varepsilon_{Nt})'$ for $t = 1, \ldots, T$. We emphasize that while we focus on linear factor models for concreteness, the t-test is valid more broadly and accommodates various types of outcome models and general forms of misspecification.

The validity of the t-test follows if we can show that the prediction model implied by the factor model satisfies the conditions required in the main text, conditional on treatment assignment D. SC methods are biased under factor models absent additional restrictions (e.g., Abadie et al., 2010; Ferman and Pinto, 2021), which motivates the proposed bias correction. We consider the following assumption.

Assumption 10. *For some random vector v, $D = f(L, v)$ and $(L, v) \perp\!\!\!\perp (F, \varepsilon)$.*

The selection mechanism $D = f(L, v)$ can be viewed as a generalization of the notion of "selection on fixed effects," which has traditionally motivated the use of DID methods, to factor models.[23] It captures the idea that the only systematic confounders are individual characteristics, similar to Imbens and Viviano (2023). We can also state Assumption 10 as $(L, D) \perp\!\!\!\perp (F, \varepsilon)$.

Define $Y_t(0) := Y_{Dt}(0)$, and denote by X_t the vector collecting $Y_{jt}(0)$ for $j \neq D$. The t-test is valid if, conditional on D, $\{(V_t(0), Z_t)\}_{t=1}^{T}$, satisfies the conditions in the main text, where $V_t(0) = Y_t(0) - \theta_t$ and $Z_t = X_t - \mathbf{1}_N \theta_t$. We establish the following result.

Lemma 4. *Suppose that the data are generated by model (18). Let Assumption 10 hold. If $\{F_t, \varepsilon_t\}_{t=1}^{T}$ is stationary, then $\{(V_t(0), Z_t)\}_{t=1}^{T}$ is stationary conditional on D.*

By Lemma 4, the validity of the t-test follows by invoking the same theory as in Section 4.1.

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[23]See, for example, Ghanem et al. (2023) for a discussion of selection mechanisms in the context of DID.

B Extensions

B.1 Time-varying predictive relationship

SC methods are based on the idea that the predictive relationship between the treated and the control units is time-invariant, such that the SC weights do not change over time. Under time-invariance, the weights can be estimated in the pre-treatment period and used to compute treatment effects in the post-treatment period. However, time-invariance of the weights is not innocuous and may be questionable in some SC applications.

The t-test accommodates certain forms of time-varying weights. Consider the following variant of model (10) with time-varying weights,

$$Y_t(0) = X_t' w_t + u_t, \quad 1 \leq t \leq T. \tag{19}$$

We assume that the time-varying weights satisfy the SC constraints, $w_t \in \mathcal{W}^{SC}$, $1 \leq t \leq T$. In Section 4.1, we assume $Y_t(0) = V_t(0) + \theta_t$ and $X_t = \mathbf{1}_N \theta_t + Z_t$, where $\{(V_t(0), Z_t)\}_{t=1}^T$ is covariance-stationary. The model (19) implies that $V_t(0) = u_t + X_t' w_t - \theta_t = u_t + Z_t' w_t$ since $w_t \in \mathcal{W}^{SC}$. Thus, as long as $\{(V_t(0), Z_t)\}_{t=1}^T$ is stationary, the results in Section 4.1 apply and the t-test remains valid with time-varying weights. Stationarity $\{(V_t(0), Z_t)\}_{t=1}^T$ is implied by stationarity of $\{(Z_t, w_t, u_t)\}_{t=1}^T$.

Time-varying coefficient models typically have additional assumptions on the nature of the variation. For example, in latent large factor models, the factors or the factor loadings can be assumed to be nonparametric functions of observed variables (e.g., Connor and Linton, 2007; Connor et al., 2012; Fan et al., 2016, 2021). We can consider a similar situation for the weights. In particular, it might be reasonable to assume that $w_t = f(Z_t, e_t)$ for some fixed unknown function $f(\cdot, \cdot)$, where $\{(Z_t, e_t, u_t)\}_{t=1}^T$ is stationary. In this case, $\{(V_t(0), Z_t)\}_{t=1}^T$ is stationary, and the theoretical results in Section 4.1 apply.

The variability in the weights over time affects the length of the confidence intervals obtained from the t-test. To see this, rewrite model (19) in terms of the pseudo-true weights w_* as

$$Y_t(0) = X_t' w_* + v_t, \quad v_t := X_t'(w_t - w_*) + u_t.$$

This shows that the variability in the weights around w_* affects the LRV of the prediction errors and thus the length of the confidence intervals.

When w_t does not vary in a stationary manner, one might question the suitability of SC methods in the application of interest. In this case, we might expect the fit of the SC model to change across time in a systemic pattern. One formal procedure to check this is the following placebo test.

1. Split the pre-treatment period into a placebo pre-treatment period, $\{1, \ldots, \widetilde{T}_0\}$, and a placebo post-treatment period $\{\widetilde{T}_0 + 1, \ldots, T_0\}$.

2. Apply the t-test, treating $\{1, \ldots, \widetilde{T}_0\}$ as the pre-treatment period and $\{\widetilde{T}_0 + 1, \ldots, T_0\}$ as the post-treatment period.

3. Reject the placebo test if zero is not in the confidence interval.

Maintaining the other assumptions underlying the t-test, a rejection indicates that the weights are time-varying in ways that render the t-test invalid. More generally, this placebo test is an omnibus specification test for the joint validity of all the assumptions underlying the t-test. We caution that while non-rejections provide evidence in favor of the applicability of the t-test, such non-rejections do not imply that the t-test valid.

In general, extrapolation is unavoidable in SC settings. Therefore, when the placebo test rejects the null of no effect, researchers need to make assumptions on how the weights change over time that allow for such extrapolation. For example, SDID accommodates a different type of time-varying weights than the t-test (via $\hat{w}_i^{\text{SDID}} \hat{\lambda}_t^{\text{SDID}}$ in (3)). In applications where the weights are time-varying in ways that are neither covered above nor by SDID, one could impose explicit models on how the weights change over time. We leave the development of such methods for future research.

B.2 Consistency of the classical synthetic control estimator

In the main text, we focus on the canonical SC estimator (6). Here we provide conditions under which the classical SC estimator of Abadie et al. (2010) satisfies the consistency requirement in Assumption 2, which implies that the t-test can also be used in conjunction with the classical SC estimator.

We start by discussing the theoretical properties of the classical SC estimator based on all pre-treatment data and discuss how to implement the cross-fitting approach underlying the t-test at the end. Consider the same setup as in the main text, but suppose that in addition to outcome data, researchers have access to a vector of covariates or predictors $Z_i \in \mathbb{R}^p$ for $i = 0, \ldots, N$.

Our results accommodate settings where the outcomes are generated by a factor model with covariates as in Abadie et al. (2010),

$$Y_{it}(0) = \theta_t + \delta_t Z_i + L_i F_t + \varepsilon_{it}, \tag{20}$$

where θ_t is a common time effect, δ_t is a time-varying coefficient, L_i is a vector of unit-specific loadings, and F_t is a vector of time-varying factors. However, importantly, our results do not

require the true data generating process to be a factor model. This feature provides useful robustness in applications.

To define the estimator and state our theoretical results, we introduce some additional notation. Define $Y_i = (Y_{i1}, \ldots, Y_{iT_0})' \in \mathbb{R}^{T_0}$, $Y_{-0} = (Y_1, \ldots, Y_N) \in \mathbb{R}^{T_0 \times N}$ and $Z_{-0} = (Z_1, \ldots, Z_N) \in \mathbb{R}^{p \times N}$.

We consider the classical SC estimator

$$\tilde{w} \in \arg\min_{w \in \mathcal{W}^{SC}} \left\| \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} - \begin{pmatrix} Y_{-0} \\ Z_{-0} \end{pmatrix} w \right\|_V^2, \tag{21}$$

where V is a positive semi-definite matrix and $\|c\|_V^2 = c'Vc$ for any $c \in \mathbb{R}^{T_0+p}$. It will be useful to partition V as

$$V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix},$$

where $V_1 \in \mathbb{R}^{T_0 \times T_0}$ and $V_2 \in \mathbb{R}^{p \times p}$. To simplify the exposition, we include all pre-treatment outcomes in estimator (21), but our results continue to hold as long as researchers use a subset of $O(T_0)$ pre-treatment outcomes. Different choices of V have been proposed (e.g., Abadie et al., 2010, 2015). Our results below allow for general V, and we do not restrict the relative magnitude of V_2 and V_1. See Remark 8 for a further discussion of the role of covariates.

The following lemma establishes the ℓ_2-consistency of $\tilde{w}$ for the pseudo-true SC weights,

$$\bar{w} = (\bar{w}_1, \ldots, \bar{w}_N)' = \arg\min_{w \in \mathcal{W}^{SC}} E \left\| \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} - \begin{pmatrix} Y_{-0} \\ Z_{-0} \end{pmatrix} w \right\|_V^2,$$

where we use $\bar{w}$ instead of w_* to denote the pseudo-true weights to distinguish them from the pseudo-true weights for the SC estimator (6) in the main text.

As in Section 4.1, we allow for a common nonstationarity. Specifically, we assume that $Y_{it}(0) = \tilde{Y}_{it}(0) + \theta_t$, where θ_t is a non-stationary process. Define $\tilde{Y}_{-0} = (\tilde{Y}_1, \ldots, \tilde{Y}_N) \in \mathbb{R}^{T_0 \times N}$ with $\tilde{Y}_i = (\tilde{Y}_{i1}, \ldots, \tilde{Y}_{iT_0})' \in \mathbb{R}^{T_0}$.

Lemma 5. *Suppose that* $\|T_0^{-1} u' V_1 \tilde{Y}_{-1} - T_0^{-1} E u' V_1 \tilde{Y}_{-1}\|_\infty = o_P(1)$ *and* $\|T_0^{-1} \tilde{Y}'_{-1} V_1 \tilde{Y}_{-1} - \Sigma_V\|_\infty = o_P(1)$, *where* $u = Y_0 = Y_{-0}\bar{w}$ *and* $\Sigma_V = E T_0^{-1} \tilde{Y}'_{-0} V_1 \tilde{Y}_{-0}$. *Suppose further that* Z_i, *for* $i \in 0, \ldots, N$, *is non-stochastic.*[24] *If* $\lambda_{\min}(\Sigma_V)$ *is bounded below by a positive constant, then* $\|\tilde{w} - \bar{w}\|_2 = o_P(1)$.

Lemma 5 allows for arbitrary misspecification and a common nonstationarity θ_t. It accommodates factor models, such as (20), but does not require the outcomes to be generated

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[24]If Z_i, $i \in 0, \ldots, N$, is stochastic, we interpret the analysis as conditional on $(Z_0, \ldots, Z_N)$.

by a factor model. Moreover, we do not require the existence of "true" SC weights that balance the individual loadings and covariates, as, for example, in Abadie et al. (2010).

Lemma 5 establishes the consistency of the classical SC estimator with covariates. Implementing the cross-fitting required for the t-test is straightforward. For each $k = 1, \ldots, K$, compute the component estimators as

$$\hat{\tau}_k = \frac{1}{T_1} \sum_{t=T_0+1}^{T} \left(Y_{0t} - \sum_{i=1}^{N} \tilde{w}_{i,(k)} Y_{it} \right) - \frac{1}{|H_k|} \sum_{t \in H_k} \left(Y_{0t} - \sum_{i=1}^{N} \tilde{w}_{i,(k)} Y_{it} \right),$$

where $\tilde{w}_{(k)} = (\tilde{w}_{1,(k)}, \ldots, \tilde{w}_{N,(k)})'$ are the weights obtained from applying the classical SC estimator to the outcome data in the subperiod $H_{(-k)} := \{1, \ldots, T_0\} \setminus H_k$ and all covariates. The estimator of the ATT and the t-statistic can be constructed as described in Section 2.2. The results in Lemma 5 imply that $\max_{1 \le k \le K} \|\tilde{w}_{(k)} - \bar{w}\|_2 = o_P(1)$ so that Assumption 2 holds with w_* replaced by $\bar{w}$, and the theoretical results in the main text imply that the t-test is valid.

Remark 8. *There are some recent discussions on the role of covariates. Although one might naturally think that including the covariates helps with the estimation accuracy of the weights, papers such as Kaul et al. (2022) show that certain ways of choosing the weights eventually lead to $V_2 = 0$, which makes covariates irrelevant. Some papers derive consistency towards a target that has no covariates even though the estimator takes into account covariates, e.g., Zhang et al. (2022). Here, we do not take a stand on this by just taking V as given. We allow for zero V_2 or large V_2. Therefore, the consistency result below applies no matter how important the covariates are.* □

B.3 Inference on the expected effect

In the main text, we propose a t-test for making inferences on the ATT, $\tau = T^{-1} \sum_{t=T_0+1}^{T} \alpha_t$. If researchers are willing restrict treatment effect heterogeneity over time and assume that $\{\alpha_t\}$ is stationary and weakly dependent, the expected treatment effect, $\tau_e = E(\alpha_t)$, is a natural alternative to the ATT (e.g., Li and Bell, 2017; Li, 2020). Here, we show how our method can be modified to make inferences on τ_e.

To motivate this modification, note that by similar arguments as in Lemma 1 and Theorem 1, we have

$$\hat{\tau}_k - \tau_e = \frac{1}{T_1} \sum_{t=T_0+1}^{T} u_t - \frac{1}{|H_k|} \sum_{t \in H_k} u_t + \frac{1}{T_1} \sum_{t=T_0+1}^{T} \tilde{\alpha}_t + o_P \left(\frac{1}{\sqrt{\min\{T_0, T_1\}}} \right).$$

where $\tilde{\alpha}_t = \alpha_t - \tau_e$. The main difference to the analysis in Section 3 is the term $T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{\alpha}_t$. This term arises because of the different centering (τ_e instead of τ) and captures randomness of the treatment effects. Due to the presence of this additional term, the statistic $\sqrt{K}(\hat{\tau} - \tau_e)/\hat{\sigma}_{\hat{\tau}}$, does not have an asymptotic t-distribution. (Note, however, that $\hat{\tau}_k$ is asymptotically unbiased for τ_e since $T_1^{-1} \sum_{t=T_0+1}^{T} u_t - |H_k|^{-1} \sum_{t \in H_k} u_t$ has mean zero under our assumptions.)

To overcome this issue, we introduce a modified debiasing approach based on also splitting up the post-treatment period. To simplify the exposition, we assume that T_1/K is an integer in the following. We split the post-treatment period into K consecutive non-overlapping blocks with T_1/K elements, $\tilde{H}_1, \ldots, \tilde{H}_K$. The estimator of the expected ATT is given by

$$\tilde{\tau} = \frac{1}{K} \sum_{k=1}^{K} \tilde{\tau}_k,$$

where

$$\tilde{\tau}_k = \frac{1}{|\tilde{H}_k|} \sum_{t \in \tilde{H}_k} \left(Y_t - X_t' \hat{w}_{(k)} \right) - \frac{1}{|H_k|} \sum_{t \in H_k} \left(Y_t - X_t' \hat{w}_{(k)} \right). \tag{22}$$

To make inferences, we again construct t-statistic based on $\{\tilde{\tau}_1, \ldots, \tilde{\tau}_K\}$. Since these estimators are based on non-overlapping blocks of data and thus asymptotically independent (see proof of Theorem 6), rescaling the denominator by $\sqrt{1 + Kr/T_1}$ is not necessary, and we can use a standard t-statistic,

$$\tilde{\mathbb{T}}_K = \frac{\sqrt{K}(\tilde{\tau} - \tau_e)}{\tilde{\sigma}_{\tilde{\tau}}},$$

where

$$\tilde{\sigma}_{\tilde{\tau}} = \sqrt{\frac{1}{K-1} \sum_{k=1}^{K} (\tilde{\tau}_k - \tilde{\tau})^2}.$$

The corresponding $(1 - \alpha)$ confidence interval for τ_e is

$$\tilde{\mathcal{I}}_K(1 - \alpha) = \left[\tilde{\tau} - t_{K-1}(1 - \alpha/2) \frac{\tilde{\sigma}_{\tilde{\tau}}}{\sqrt{K}}, \ \hat{\tau} + t_{K-1}(1 - \alpha/2) \frac{\tilde{\sigma}_{\tilde{\tau}}}{\sqrt{K}} \right].$$

The following theorem provides a formal justification of the modified inference method.

Theorem 6. *Let the assumptions in Theorem 1 hold, except that we modify Assumption 3 by replacing (X_t, u_t) with (X_t, u_t, α_t). Then (i) $\tilde{\mathbb{T}}_K \xrightarrow{d} t_{K-1}$ and (ii) $P(\tau_e \in \tilde{\mathcal{I}}_K(1 - \alpha)) \to 1 - \alpha$.*

The proof of Theorem (6) shows that the component estimators $\tilde{\tau}_1, \ldots, \tilde{\tau}_K$ are asymptotically independent due to the sample splitting in the post-treatment period, so that one can use a standard t-statistic. This result has two important practical implications. First, the

t-test based on $\tilde{\mathbb{T}}_K$ is robust against heterogeneity in the variances of the component esti-mators (Bakirov and Székely, 2005; Ibragimov and Müller, 2010) and thus is robust against certain types of nonstationarity in $\{u_t\}$. Second, the sign-based randomization test of Canay et al. (2017) constitutes a natural alternative to the t-test, provided that K is large enough. The randomization test is particularly appealing when stationarity of $\{u_t\}$ is questionable: it is robust to heterogeneity in the variances across component estimators and more powerful than the t-test in such settings (Canay et al., 2017).

C   Robustness of the t-test

The results in Bakirov and Székely (2005) and Ibragimov and Müller (2010) show that the standard t-test is robust against heterogeneity in the variances of the component estimators for $\alpha \leq 2\Phi(-3) \approx 0.083$. In our context, such heterogeneity arises when $\{u_t\}$ is not station-ary, and the LRV varies across blocks within the pre-treatment period and/or between the pre-treatment and the post-treatment period.

The existing robustness results do not directly apply to the t-test described in Section 2.2 because the component estimators are not asymptotically independent. Nevertheless, as we discuss below, the proposed t-test inherits certain robustness properties from the standard t-test.

Suppose that $\{u_t\}$ is non-stationary with variance changing across blocks and between the pre-treatment and post-treatment period so that

$$\sqrt{\min\{T_0, T_1\}} \begin{pmatrix} \hat{\tau}_1 - \tau \\ \vdots \\ \hat{\tau}_K - \tau \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \sqrt{\min\{c_0, 1\}}\sigma_0\xi_0 - \sqrt{g_{c_0,K}}\sigma_1\xi_1 \\ \vdots \\ \sqrt{\min\{c_0, 1\}}\sigma_0\xi_0 - \sqrt{g_{c_0,K}}\sigma_K\xi_K \end{pmatrix},$$

where $\xi_0, \ldots, \xi_K$ are independent $N(0,1)$ random variables and $\sigma_0, \ldots, \sigma_K$ are the block-specific LRVs. By the continuous mapping theorem, $\mathbb{T}_K \xrightarrow{d} \mathcal{T}_K$, where

$$\mathcal{T}_K = \frac{\sqrt{\frac{K}{g_{c_0,K}(1+\min\{c_0,K\})}} \left(\sqrt{\min\{c_0, 1\}}\sigma_0\xi_0 - \sqrt{g_{c_0,K}}K^{-1}\sum_{k=1}^{K}\sigma_k\xi_k \right)}{\sqrt{(K-1)^{-1}\sum_{k=1}^{K}(\sigma_k\xi_k - K^{-1}\sum_{k=1}^{K}\sigma_k\xi_k)^2}}$$

$$= \frac{\sqrt{\frac{K\min\{c_0,1\}}{g_{c_0,K}(1+\min\{c_0,K\})}}\sigma_0\xi_0}{\sqrt{(K-1)^{-1}\sum_{k=1}^{K}(\sigma_k\xi_k - K^{-1}\sum_{k=1}^{K}\sigma_k\xi_k)^2}} - \frac{1}{\sqrt{1+\min\{c_0,K\}}}\mathbf{T}_K, \quad (23)$$

and $\mathbf{T}_K$ is the standard t-statistic, which is robust to variance heterogeneity.

The decomposition (23) shows that, while the proposed t-test inherits some robustness from the standard t-test due to the second term, it is not fully robust against heterogeneity in general.

Importantly, however, the t-test is robust under different types of restrictions on the LRVs $\sigma_0, \ldots, \sigma_K$. A leading example is when $\{u_t\}$ is non-stationary due to a structural break between the pre-treatment and post-treatment period, so that $\sigma_k = \sigma_{\text{pre}} > 0$ for $k \geq 1$ and $\sigma_0 = \sigma_{\text{post}} > 0$ with $\sigma_{\text{pre}} \neq \sigma_{\text{post}}$. In this case, we have that

$$\mathcal{T}_K = \frac{\sqrt{\frac{K}{g_{c_0,K}(1+\min\{c_0,K\})}}\left(\sqrt{\min\{c_0,1\}}\frac{\sigma_{\text{post}}^2}{\sigma_{\text{pre}}^2}\xi_0 - \sqrt{g_{c_0,K}}\bar{\xi}\right)}{\sqrt{(K-1)^{-1}\sum_{k=1}^{K}(\xi_k - \bar{\xi}_k)^2}}.$$

Note that the numerator has a $N(0, \sigma_{\text{num}}^2)$ distribution, where

$$\sigma_{\text{num}}^2 = \frac{K}{g_{c_0,K}(1+\min\{c_0,K\})} \times \left(\min\{c_0,1\}\frac{\sigma_{\text{post}}^2}{\sigma_{\text{pre}}^2} + g_{c_0,K}K^{-1}\right) = \begin{cases} \frac{1+\frac{\sigma_{\text{post}}^2}{\sigma_{\text{pre}}^2}c_0}{1+c_0} & \text{if } 1 \leq c_0 \leq K \\ \frac{1+\frac{\sigma_{\text{post}}^2}{\sigma_{\text{pre}}^2}c_0}{1+c_0} & \text{if } c_0 < 1 \\ \frac{1+\frac{\sigma_{\text{post}}^2}{\sigma_{\text{pre}}^2}K}{1+K} & \text{if } c_0 > K. \end{cases}$$

Note that $\sigma_{\text{num}}^2 \leq 1$ if $\sigma_{\text{post}}^2/\sigma_{\text{pre}}^2 \leq 1$. Therefore, by the independence of the numerator and denominator, we can condition on the denominator to show that the t-test remains valid in this case:

$$P(|\mathcal{T}_K| > t_{K-1}(1-\alpha/2)) = E\left(E\left(1\{|\mathcal{T}_K| > t_{K-1}(1-\alpha/2)\} \mid \sqrt{(K-1)^{-1}\sum_{k=1}^{K}(\xi_k-\bar{\xi}_k)^2}\right)\right)$$

$$\leq E\left(E\left(1\{|\mathbf{T}_K| > t_{K-1}(1-\alpha/2)\} \mid \sqrt{(K-1)^{-1}\sum_{k=1}^{K}(\xi_k-\bar{\xi}_k)^2}\right)\right)$$

$$= P(|\mathbf{T}_K| > t_{K-1}(1-\alpha/2)) \quad \text{for} \quad \frac{\sigma_{\text{post}}^2}{\sigma_{\text{pre}}^2} \leq 1.$$

The above derivation shows that the t-test remains valid when $\{u_t\}$ is non-stationary due to a decrease in the LRV between the pre- and post-treatment period ($\sigma_{\text{post}}/\sigma_{\text{pre}} \leq 1$). A similar argument shows that the t-test will not be valid in general when the LRV increases so that $\sigma_{\text{post}}/\sigma_{\text{pre}} > 1$.

The above derivation also suggests a simple modification of the t-statistic for applications where researchers are concerned about an increase in the LRV. Suppose that $\sigma_{\text{post}}^2/\sigma_{\text{pre}}^2 \leq B_{\sigma_{\text{post}}^2/\sigma_{\text{pre}}^2}$, which implies an upper bound on the variance of the numerator,

$$\sigma_{\text{num}}^2 \leq B_{\sigma_{\text{num}}^2} = \frac{1 + B_{\sigma_{\text{post}}^2/\sigma_{\text{pre}}^2}c_0}{1+c_0}1\{c_0 \leq K\} + \frac{1 + B_{\sigma_{\text{post}}^2/\sigma_{\text{pre}}^2}K}{1+K}1\{c_0 > K\}.$$

Then, the following modified t-statistic yields valid inferences:

$$\mathbb{T}_K = \frac{\sqrt{K}\,(\hat{\tau} - \tau)}{\sqrt{1 + \frac{Kr}{T_1}}\sqrt{B_{\sigma^2_{\text{num}}}}\sqrt{\frac{1}{K-1}\sum_{k=1}^{K}(\hat{\tau}_k - \hat{\tau})^2}}.$$

D   Higher-order improvements

Suppose that we observe $\{y_t\}_{t=1}^{T}$ from the following Gaussian location model:

$$y_t = \beta + u_t, \tag{24}$$

where u_t is a mean zero covariance-stationary Gaussian process with $\sum_{h=-\infty}^{\infty} h^2 \gamma(h) < \infty$ and $\gamma(h) = E u_t u_{t-h}$. This model has been considered for example by Jansson (2004) and Sun et al. (2008, Assumption 3). Sun et al. (2008) show that the usual consistent estimators using Bartlett kernel for the LRV (e.g., Newey-West estimator) would generate a test that has size distortion at least $O(T^{-1/2})$. They also show that the fixed-b approach would have size distortion of order $O(T^{-1})$. These results are the typical explanation for why the fixed-b approach is more accurate than the classical LRV estimation approach. We now show that in this setting our self-normalization approach also enjoys the higher-order improvement as it has size distortion of order $O(T^{-1})$.

We first outline the inference procedure in the Gaussian location model in (24). Let $K \geq 2$ be a fixed integer and define $G = T/K$. For simplicity, we assume that G is an integer. Then we construct blocks $\{H_k\}_{k=1}^{K}$ with $H_k = \{(k-1)G + 1, \ldots, kG\}$. Let $\hat{\beta}_k = G^{-1}\sum_{t \in H_k} y_t$ for $1 \leq k \leq K$ and $\bar{\beta} = K^{-1}\sum_{k=1}^{K}\hat{\beta}_k$. For testing $H_0 : \beta = 0$, we define the test statistic

$$\mathbf{T}_K = \frac{\sqrt{K}\bar{\beta}}{\sqrt{(K-1)^{-1}\sum_{k=1}^{K}(\hat{\beta}_k - \bar{\beta})^2}}.$$

The critical value is $t_{K-1}(1 - \alpha/2)$.

Theorem 7. *Consider the model in (24). Suppose that $H_0 : \beta = 0$ holds. Then*

$$|P\left(|\mathbf{T}_K| > t_{K-1}(1 - \alpha/2)\right) - \alpha| = O(T^{-1}).$$

By Theorem 7, the "cross-fitted" self-normalized t-test also has size distortion $O(T^{-1})$ in the Gaussian location model (24). Although it is quite difficult to derive the higher-order asymptotics outside the Gaussian location model, Theorem 7 suggests that, for size considerations, the cross-fitted self-normalized test is expected to have similar properties as fixed-b methods. However, in high-dimensional settings, we are not aware of any existing results that would allow for establishing the validity of the fixed-b approach in our context.

E Calculations expected length

The length of the confidence interval is

$$L = 2t_{K-1}(1 - \alpha/2)\frac{1}{\sqrt{K}}\sqrt{1 + \frac{Kr}{T_1}}\sqrt{\frac{1}{K-1}\sum_{k=1}^{K}(\hat{\tau}_k - \hat{\tau})^2}.$$

Consider

$$\sqrt{\min\{T_0, T_1\}}L = 2t_{K-1}(1 - \alpha/2)\frac{1}{\sqrt{K}\sqrt{K-1}}\sqrt{1 + \frac{Kr}{T_1}}\sqrt{\min\{T_0, T_1\}\sum_{k=1}^{K}(\hat{\tau}_k - \hat{\tau})^2}.$$

To derive the limiting distribution for fixed K, note that

$$\min\{T_0, T_1\}\sum_{k=1}^{K}(\hat{\tau}_k - \hat{\tau})^2 = \sum_{k=1}^{K}\left(\sqrt{\min\{T_0, T_1\}}(\hat{\tau}_k - \tau) - \frac{1}{K}\sum_{k=1}^{K}\sqrt{\min\{T_0, T_1\}}(\hat{\tau}_k - \tau)\right)^2$$

and that $Kr/T_1 \to \min\{c_0, K\}$ as $T_0, T_1 \to \infty$.

Therefore, by Theorem 1 and the continuous mapping theorem, $\sqrt{\min\{T_0, T_1\}}L \xrightarrow{d} \mathcal{L}$, where

$$
\begin{aligned}
\mathcal{L} &= 2\sigma t_{K-1}(1 - \alpha/2)\frac{1}{\sqrt{K}\sqrt{K-1}}\sqrt{1 + \min\{c_0, K\}} \\
&\quad \times \sqrt{\sum_{k=1}^{K}\left(\sqrt{\min\{c_0, 1\}}\xi_0 - \sqrt{g_{c_0,K}}\xi_k - \frac{1}{K}\sum_{k=1}^{K}\sqrt{\min\{c_0, 1\}}\xi_0 - \sqrt{g_{c_0,K}}\xi_k\right)^2} \\
&= 2\sigma t_{K-1}(1 - \alpha/2)\frac{1}{\sqrt{K}\sqrt{K-1}}\sqrt{1 + \min\{c_0, K\}}\sqrt{g_{c_0,K}}\sqrt{\sum_{k=1}^{K}\left(\xi_k - \frac{1}{K}\sum_{k=1}^{K}\xi_k\right)^2}
\end{aligned}
$$

Thus, it remains to compute the expectation of

$$\sqrt{\sum_{k=1}^{K}\left(\xi_k - \frac{1}{K}\sum_{k=1}^{K}\xi_k\right)^2}$$

Because

$$\sum_{k=1}^{K}\left(\xi_k - \frac{1}{K}\sum_{k=1}^{K}\xi_k\right)^2 \sim \chi_{K-1}^2,$$

so that

$$\sqrt{\sum_{k=1}^{K}\left(\xi_k - \frac{1}{K}\sum_{k=1}^{K}\xi_k\right)^2} \sim \chi_{K-1},$$

where χ_{K-1} denotes a chi-distribution with $K-1$ degrees of freedom. Therefore,

$$E\left(\sqrt{\sum_{k=1}^{K}\left(\xi_k - \frac{1}{K}\sum_{k=1}^{K}\xi_k\right)^2}\right) = \sqrt{2}\frac{\Gamma(K/2)}{\Gamma((K-1)/2)},$$

where $\Gamma(\cdot)$ denotes the Gamma function, and

$$E(\mathcal{L}) = 2\sigma t_{K-1}(1-\alpha/2)\frac{1}{\sqrt{K}\sqrt{K-1}}\sqrt{1+\min\{c_0, K\}}\sqrt{g_{c_0,K}}\sqrt{2}\frac{\Gamma(K/2)}{\Gamma((K-1)/2)}.$$

Next, we provide an approximation of the limit of $E(\mathcal{L})$ as $K \to \infty$. Since $K \to \infty$, assuming that $c_0 < \infty$, we have that for large K,

$$g_{c_0,K} = K\mathbf{1}\{c_0 < 1\} + (K/c_0)\mathbf{1}\{1 \le c_0 \le K\} = \min\{c_0^{-1}, 1\}K.$$

Note further that we take $T_1 \to \infty$ before taking $K \to \infty$ so that $1 + Kr/T_1 = 1$. To understand what happens when $K \to \infty$, we use Stirling's approximation of the Gamma function: $\Gamma(n+1) = (1+o(1)) \times \sqrt{2\pi n}(n/e)^n$ as $n \to \infty$. This means that

$$\frac{\Gamma(n+3/2)}{\Gamma(n+1)} = (1+o(1)) \cdot \frac{((n+1/2)/e)^{n+1/2}}{(n/e)^n} = (1+o(1)) \cdot \left(\frac{n+1/2}{n}\right)^n \cdot \left(\frac{n+1/2}{e}\right)^{1/2}$$

$$= (1+o(1)) \cdot e^{1/2}\left(\frac{n+1/2}{e}\right)^{1/2} = (1+o(1)) \cdot \sqrt{n}.$$

Therefore, and using that $t_{K-1}(1-\alpha) \to \Phi^{-1}(1-\alpha/2)$ as $K \to \infty$, we have the following approximation as $K \to \infty$,

$$E(\mathcal{L}) = 2\sigma\Phi^{-1}(1-\alpha/2)\frac{1}{\sqrt{K}\sqrt{K-1}}\sqrt{g_{c_0,K}}\sqrt{2}\frac{\Gamma(K/2)}{\Gamma((K-1)/2)}\sqrt{1+\min\{c_0, K\}}$$

$$= (1+o(1)) \cdot 2\sigma\Phi^{-1}(1-\alpha/2)\frac{1}{\sqrt{K}\sqrt{K-1}}\sqrt{\min\{c_0^{-1}, 1\}K}\sqrt{2} \cdot \sqrt{K/2}\sqrt{1+c_0}$$

$$= (1+o(1)) \cdot 2\sigma\Phi^{-1}(1-\alpha/2)\sqrt{\min\{c_0^{-1}, 1\}}\sqrt{1+c_0}.$$

F   Verification of Assumption 6

Here we illustrate the verification of Assumption 6 based on two leading examples.

Example 1 (Unit root). Assume that $\xi_t = \sum_{i=1}^{t}v_t$ with a mean-zero stationary process v_t. Then by the functional central limit theorem, we can show that $T^{-1/2}\xi_{[rT]}$ tends to $B(r)$, where $B(\cdot)$ is a standard Brownian motion. Thus, the continuous mapping theorem and the condition of $T_1 \ll T_0$ imply $T_1^{-1}T^{-1/2}\sum_{t=T_0+1}^{T}\xi_t \xrightarrow{d} B(1)$ and $T^{-2}\sum_{t\in H_{(-k)}}\xi_t^2 = T^{-2}\sum_{t=1}^{T_0}\xi_t^2 - T^{-2}\sum_{t\in H_{(k)}}\xi_t^2 \xrightarrow{d} \int_0^1 B^2(s)ds$. This verifies the first part of Assumption 6. The second part follows by a similar calculation. $\qquad\square$

Example 2 (Polynomial trend). Assume that $\xi_t = \sum_{k=1}^{M} c_k t^k$ with constants $c_1, \dots, c_M$, where $M \geq 1$ is fixed. ($M = 1$ corresponds to a linear trend and $M = 2$ represents a quadratic trend.) Recall the elementary relation of $(\sum_{i=1}^{n} i^\gamma)/(n^{1+\gamma}/(1+\gamma)) \to 1$ as $n \to \infty$ for any fixed $\gamma > 0$. By $T_1 \ll T_0$, the first part of Assumption 6 holds:

$$\frac{\sum_{t=T_0+1}^{T} \xi_t}{\sqrt{\sum_{t \in H_{(-k)}} \xi_t^2}} = (1 + o(1)) \cdot \frac{(1+M)^{-1} \left[(1 + T_1/T_0)^{M+1} - 1 \right] T_0^{M+1}}{\sqrt{(1+2M)^{-1} T_0^{2M+1}}}$$

$$= O(T_1 T^{-1/2}).$$

The second part of Assumption 6 follows by a similar calculation. $\square$

G Proofs

We will sometimes write Y_t instead of $Y_t(0)$ to simplify the exposition.

G.1 Proof of Lemma 1

Let $\mu = E(X_t)$. Notice that for $T_0 + 1 \leq t \leq T$,

$$Y_t - X_t' \hat{w}_{(k)} = \alpha_t + u_t - X_t' \Delta_{(k)} = \alpha_t + u_t - \mu' \Delta_{(k)} - \tilde{X}_t' \Delta_{(k)}$$

and for $t \in H_k$,

$$Y_t(0) - X_t' \hat{w}_{(k)} = u_t - X_t' \Delta_{(k)} = u_t - \mu' \Delta_{(k)} - \tilde{X}_t' \Delta_{(k)}.$$

Therefore,

$$\hat{\tau}_k - \tau = T_1^{-1} \sum_{t=T_0+1}^{T} (Y_t - X_t' \hat{w}_{(k)}) - \frac{1}{|H_k|} \sum_{t \in H_k} (Y_t - X_t' \hat{w}_{(k)}) - \tau$$

$$= T_1^{-1} \sum_{t=T_0+1}^{T} (\alpha_t + u_t - \mu' \Delta_{(k)} - \tilde{X}_t' \Delta_{(k)}) - \frac{1}{|H_k|} \sum_{t \in H_k} (u_t - \mu' \Delta_{(k)} - \tilde{X}_t' \Delta_{(k)}) - \tau$$

$$= T_1^{-1} \sum_{t=T_0+1}^{T} u_t - \frac{1}{|H_k|} \sum_{t \in H_k} u_t + \frac{1}{|H_k|} \sum_{t \in H_k} \tilde{X}_t' \Delta_{(k)} - T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{X}_t' \Delta_{(k)}.$$

The proof is complete. $\square$

G.2 Proof of Lemma 2

We prove the results for the following more general estimator:

$$\hat{w}_{(k)} \in \arg\min_{w \in \mathcal{W}} \sum_{t \in H_{(-k)}} (Y_t - X_t' w)^2, \tag{25}$$

15

where $\mathcal{W}$ is a subset of $\{w : \|w\|_1 \leq Q\}$ with $Q = O(1)$. For the canonical SC estimator, we see that $\mathcal{W} = \{w = (w_1, \ldots, w_N)' : w_i \geq 0, \sum_{i=1}^N w_i = 1\}$ and thus $Q = 1$.

For simplicity, we write $\hat{w} = \hat{w}_{(k)}$, $\hat{\mu} = \hat{\mu}_{(-k)}$ and $\hat{\Sigma} = \hat{\Sigma}_{(-k)}$. Let $\xi = (\hat{\mu} - \hat{\Sigma}w_*) - (\mu - \Sigma w_*)$.

Since $\|w_*\|_\infty \leq 2Q$, we have that $\|\xi\|_\infty \leq \|\hat{\mu} - \mu\|_\infty + \|(\hat{\Sigma} - \Sigma)w_*\|_\infty \leq \|\hat{\mu} - \mu\|_\infty + 2Q\|\hat{\Sigma} - \Sigma\|_\infty$. Thus, by assumption, $\|\xi\|_\infty = o_P(1)$.

We rewrite

$$w_* = \arg\min_v \ v'\Sigma v - 2\mu'v \qquad s.t. \qquad v \in \mathcal{W}$$

and

$$\hat{w} \in \arg\min_v \ v'\hat{\Sigma}v - 2\hat{\mu}'v \qquad s.t. \qquad v \in \mathcal{W}$$

Let $\Delta = \hat{w} - w_*$. For any $\lambda \in [0, 1]$, define $w_\lambda = w_* + \lambda\Delta$. Then by the definition of w_*, we have that $w_\lambda'\Sigma w_\lambda - 2\mu'w_\lambda \geq w_*'\Sigma w_* - 2\mu'w_*$, which means

$$\lambda^2 \Delta'\Sigma\Delta \geq 2\lambda(\mu - \Sigma w_*)'\Delta.$$

Thus, for any $\lambda \in (0, 1)$, we have that $\lambda\Delta'\Sigma\Delta \geq 2(\mu - \Sigma w_*)'\Delta$. Since this holds for any $\lambda \in (0, 1)$, we have that

$$(\mu - \Sigma w_*)'\Delta \leq 0. \tag{26}$$

Now by definition, we have that

$$\hat{w}'\hat{\Sigma}\hat{w} - 2\hat{\mu}'\hat{w} \leq w_*'\hat{\Sigma}w_* - 2\hat{\mu}'w_*.$$

It follows that

$$\Delta'\hat{\Sigma}\Delta \leq 2(\hat{\mu} - \hat{\Sigma}w_*)'\Delta.$$

Notice that

$$(\hat{\mu} - \hat{\Sigma}w_*)'\Delta = (\mu - \Sigma w_*)'\Delta + \xi'\Delta \overset{(i)}{\leq} \xi'\Delta \leq \|\xi\|_\infty\|\Delta\|_1 \leq 2\|\xi\|_\infty Q,$$

where (i) holds by (26). The above two displays imply that

$$\Delta'\hat{\Sigma}\Delta \leq 2\|\xi\|_\infty Q.$$

Therefore,

$$\begin{aligned} 2\|\xi\|_\infty Q \geq \Delta'\Sigma\Delta + \Delta'(\hat{\Sigma} - \Sigma)\Delta &\geq \Delta'\Sigma\Delta - \|\hat{\Sigma} - \Sigma\|_\infty\|\Delta\|_1^2 \\ &\geq c\|\Delta\|_2^2 - 4\|\hat{\Sigma} - \Sigma\|_\infty Q^2. \end{aligned}$$

The desired result follows by recalling $\|\xi\|_\infty \leq \|\hat{\mu} - \mu\|_\infty + 2Q\|\hat{\Sigma} - \Sigma\|_\infty$. $\qquad\qquad\square$

G.3 Proof of Lemma 3

By Equations (15) and (16) the result follows from the same arguments as in the proof of Lemma 2 with $\mathcal{W} = \mathcal{W}^{SC}$. □

G.4 Proof of Lemma 4

Note that

$$
\begin{aligned}
Y_t(0) &= \sum_{i=0}^{D} \mathbf{1}\{D = i\} Y_{it}(0) \\
&= \sum_{i=0}^{D} \mathbf{1}\{D = i\}(\theta_t + L_i' F_t + \varepsilon_{it}) \\
&= \theta_t + F_t'\left(\sum_{i=0}^{D} \mathbf{1}\{D = i\} L_i\right) + \sum_{i=0}^{D} \mathbf{1}\{D = i\}\varepsilon_{it} \\
&= V_t(0) + \theta_t,
\end{aligned}
$$

where $V_t(0) = F_t'\left(\sum_{i=0}^{D} \mathbf{1}\{D = i\} L_i\right) + \sum_{i=0}^{D} \mathbf{1}\{D = i\}\varepsilon_{it}$. For $j \neq D$, we have that

$$
Y_{jt}(0) = \theta_t + L_j' F_t + \varepsilon_{jt} = \theta_t + Z_{jt},
$$

where $Z_{jt} = L_j' F_t + \varepsilon_{jt}$.

Next, note that Assumption 10 implies $(L, D) \perp\!\!\!\perp (F, \varepsilon)$. Let $\mathcal{F}$ be σ-algebra generated by (L, D). Since (L, D) is independent of (F, ε) and (F_t, ε_t) is stationary, we have that conditional on $\mathcal{F}$, the following is stationary:

$$
\left[F_t'\left(\sum_{i=0}^{D} \mathbf{1}\{D = i\} L_i\right) + \sum_{i=0}^{D} \mathbf{1}\{D = i\}\varepsilon_{it}, \{L_j' F_t + \varepsilon_{jt}\}_{j \neq D} \right].
$$

Stationarity conditional on (L, D) then implies stationarity conditional on D since L is time-invariant.

G.5 Proof of Lemma 5

Let $\Delta = \tilde{w} - \bar{w}$ and $e = Z_0 - Z_{-0}\bar{w}$. Since $\bar{w} \in \mathcal{W}^{SC}$, the definition of $\tilde{w}$ implies that

$$
\left\| \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} - \begin{pmatrix} Y_{-0} \\ Z_{-0} \end{pmatrix} \tilde{w} \right\|_V^2 \leq \left\| \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} - \begin{pmatrix} Y_{-0} \\ Z_{-0} \end{pmatrix} \bar{w} \right\|_V^2. \tag{27}
$$

The right-hand side is $\left\| \begin{pmatrix} u \\ e \end{pmatrix} \right\|_V^2 = u'V_1 u + e'V_2 e$. For the left-hand side, we have

$$\begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} - \begin{pmatrix} Y_{-0} \\ Z_{-0} \end{pmatrix} \bar{w} = \begin{pmatrix} Y_{-0}\bar{w} + u \\ Z_{-0}\bar{w} + e \end{pmatrix} - \begin{pmatrix} Y_{-0} \\ Z_{-0} \end{pmatrix} \hat{w} = \begin{pmatrix} u - Y_{-0}\Delta \\ e - Z_{-0}\Delta \end{pmatrix} \overset{\text{(i)}}{=} \begin{pmatrix} u - \tilde{Y}_{-0}\Delta \\ e - Z_{-0}\Delta \end{pmatrix}.$$

where (i) follows by $Y_{-0}\Delta = (\delta \mathbf{1}_N' + \tilde{Y}_{-0})\Delta = \tilde{Y}_{-0}\Delta$ since $\mathbf{1}_N'\Delta = \mathbf{1}_N'(\tilde{w} - \bar{w}) = 1 - 1 = 0$, where $\delta = (\delta_1, \ldots, \delta_{T_0})'$. Thus, (27) implies that

$$\left(u - \tilde{Y}_{-0}\Delta \right)' V_1 \left(u - \tilde{Y}_{-0}\Delta \right) + (e - Z_{-0}\Delta)' V_2 (e - Z_{-0}\Delta) \le u'V_1 u + e'V_2 e.$$

Rearranging the terms, we obtain

$$\Delta' \left(T_0^{-1}\tilde{Y}_{-0}'V_1\tilde{Y}_{-0} \right) \Delta + \Delta' \left(T_0^{-1}Z_{-0}'V_2 Z_{-0} \right) \Delta \le 2 \left(T_0^{-1}u'V_1\tilde{Y}_{-0} \right) \Delta + 2 \left(T_0^{-1}e'V_2 Z_{-0} \right) \Delta. \tag{28}$$

We observe that for any w,

$$T_0^{-1} E \left\| \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} - \begin{pmatrix} Y_{-0} \\ Z_{-0} \end{pmatrix} w \right\|_V^2$$

$$= T_0^{-1} E \left\| \begin{pmatrix} \tilde{Y}_0 \\ Z_0 \end{pmatrix} - \begin{pmatrix} \tilde{Y}_{-0} \\ Z_{-0} \end{pmatrix} w \right\|_V^2$$

$$= T_0^{-1} E \| \tilde{Y}_0 - \tilde{Y}_{-0}w \|_{V_1}^2 + T_0^{-1}\| Z_0 - Z_{-0}w \|_{V_2}^2$$

$$= T_0^{-1} E \| \tilde{Y}_0 \|_{V_1}^2 - 2 \left(T_0^{-1} E \tilde{Y}_0'V_1\tilde{Y}_{-0} \right) w + w' \left(T_0^{-1} E \tilde{Y}_{-0}'V_1\tilde{Y}_{-0} \right) w + T_0^{-1}\| Z_0 - Z_{-0}w \|_{V_2}^2.$$

Let $w_\lambda = \bar{w} + \lambda\Delta$. Notice that $w_\lambda \in \mathcal{W}^{SC}$ for any $\lambda \in [0, 1]$. Thus, by the definition of $\bar{w}$, we have that

$$\frac{\partial}{\partial \lambda} \left(-2 \left(T_0^{-1} E \tilde{Y}_0'V_1\tilde{Y}_{-1} \right) w_\lambda + w_\lambda' \left(T_0^{-1} E \tilde{Y}_{-0}'V_1\tilde{Y}_{-0} \right) w_\lambda + T_0^{-1}\| Z_0 - Z_{-0}w_\lambda \|_{V_2}^2 \right) \Bigg|_{\lambda=0} \ge 0.$$

This means that

$$-2 \left(T_0^{-1} E \tilde{Y}_0'V_1\tilde{Y}_{-0} \right) \Delta + 2\bar{w}' \left(T_0^{-1} E \tilde{Y}_{-0}'V_1\tilde{Y}_{-0} \right) \Delta - 2T_0^{-1}\Delta Z_{-0}'V_2(Z_0 - Z_{-0}\bar{w}) \ge 0.$$

In other words,

$$\left(T_0^{-1} E u'V_1\tilde{Y}_{-0} \right) \Delta + T_0^{-1}e'V_2 Z_{-0}\Delta \le 0. \tag{29}$$

Hence, by the assumption of $\| T_0^{-1}u'V_1\tilde{Y}_{-0} - T_0^{-1} E u'V_1\tilde{Y}_{-0} \|_\infty = o_P(1)$, we have that

$$\left(T_0^{-1}u'V_1\tilde{Y}_{-0} \right) \Delta + T_0^{-1}e'V_2 Z_{-0}\Delta$$

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$$= \left(T_0^{-1} E u' V_1 \tilde{Y}_{-0}\right) \Delta + T_0^{-1} e' V_2 Z_{-0} \Delta + \left(T_0^{-1} u' V_1 \tilde{Y}_{-0} - T_0^{-1} E u' V_1 \tilde{Y}_{-0}\right) \Delta$$

$$\overset{(i)}{\leq} \left(T_0^{-1} u' V_1 \tilde{Y}_{-0} - T_0^{-1} E u' V_1 \tilde{Y}_{-0}\right) \Delta$$

$$\leq \|T_0^{-1} u' V_1 \tilde{Y}_{-0} - T_0^{-1} E u' V_1 \tilde{Y}_{-0}\|_\infty \|\Delta\|_1 \overset{(ii)}{\leq} o_P(1),$$

where (i) follows by (29) and (ii) follows by $\|\Delta\|_1 \leq \|\tilde{w}\|_1 + \|\bar{w}\|_1 = 2$. This and (28) imply

$$\Delta' \left(T_0^{-1} \tilde{Y}'_{-0} V_1 \tilde{Y}_{-0}\right) \Delta + \Delta' \left(T_0^{-1} Z'_{-0} V_2 Z_{-0}\right) \Delta \leq o_P(1).$$

On the other hand, we have

$$\Delta' \left(T_0^{-1} \tilde{Y}'_{-0} V_1 \tilde{Y}_{-0}\right) \Delta$$

$$= \Delta' \left(T_0^{-1} E \tilde{Y}'_{-0} V_1 \tilde{Y}_{-0}\right) \Delta + \Delta' \left(T_0^{-1} \tilde{Y}'_{-0} V_1 \tilde{Y}_{-0} - T_0^{-1} E \tilde{Y}'_{-0} V_1 \tilde{Y}_{-0}\right) \Delta$$

$$\geq \Delta' \left(T_0^{-1} E \tilde{Y}'_{-0} V_1 \tilde{Y}_{-0}\right) \Delta - \|T_0^{-1} \tilde{Y}'_{-0} V_1 \tilde{Y}_{-0} - T_0^{-1} E \tilde{Y}'_{-0} V_1 \tilde{Y}_{-0}\|_\infty \|\Delta\|_1^2$$

$$= \Delta' \left(T_0^{-1} E \tilde{Y}'_{-0} V_1 \tilde{Y}_{-0}\right) \Delta - o_P(1).$$

The above two displays imply

$$\Delta' \left(T_0^{-1} E \tilde{Y}'_{-0} V_1 \tilde{Y}_{-0}\right) \Delta + \Delta' \left(T_0^{-1} Z'_{-0} V_2 Z_{-0}\right) \Delta = o_P(1).$$

Since $\Delta' \left(T_0^{-1} Z'_{-0} V_2 Z_{-0}\right) \Delta \geq 0$ and the smallest eigenvalue of $T_0^{-1} E \tilde{Y}'_{-0} V_1 \tilde{Y}_{-0}$ is bounded away from zero, we have $\|\Delta\|_2 = o_P(1)$.

G.6 Proof of Theorem 1

By Lemma 1, we have that

$$\left| \hat{\tau}_k - \tau - \left(\frac{1}{T_1} \sum_{t=T_0+1}^{T} u_t - \frac{1}{|H_k|} \sum_{t \in H_k} u_t \right) \right| \leq \left| \frac{1}{|H_k|} \sum_{t \in H_k} \tilde{X}'_t \Delta_{(k)} \right| + \left| \frac{1}{T_1} \sum_{t=T_0+1}^{T} \tilde{X}'_t \Delta_{(k)} \right|, \quad (30)$$

where $\tilde{X}_t = X_t - E(X_t)$ and $\Delta_{(k)} = \hat{w}_{(k)} - w_*$.[25] We now bound the two terms on the right-hand side of (30).

Fix $k \in \{1, \ldots, K\}$. Define B_k to be the "two-sided buffer", i.e., the set that contains the smallest γ_T numbers and the largest γ_T numbers in $H_k = \{(k-1)r + 1, \ldots, kr\}$ for $1 \leq k \leq K$. Also define $A_k = H_k \backslash B_k$, i.e., $A_k = \{t : (k-1)r + 1 + \gamma_T + 1 \leq t \leq kr - \gamma_T\}$. Thus,

$$\sum_{t \in H_k} \tilde{X}'_t \Delta_{(k)} = \sum_{t \in A_k} \tilde{X}'_t \Delta_{(k)} + \sum_{t \in B_k} \tilde{X}'_t \Delta_{(k)}.$$

---

[25]We note that working with $\tilde{X}_t$ instead of X_t is key for our theoretical argument since $E(\tilde{X}_t) = 0$.

The second term can be bounded by

$$\left| \sum_{t \in B_k} \tilde{X}'_t \Delta_{(k)} \right| \leq \max_{1 \leq t \leq T_0} \|\tilde{X}_t\|_\infty \|\Delta_{(k)}\|_1 |B_k| = 2\gamma_T \max_{1 \leq t \leq T_0} \|\tilde{X}_t\|_\infty \|\Delta_{(k)}\|_1.$$

Thus,

$$P\left(\left| \sum_{t \in B_k} \tilde{X}'_t \Delta_{(k)} \right| \leq 2\rho_T \gamma_T \|\Delta_{(k)}\|_1 \right) \to 1. \tag{31}$$

On the other hand, we use Berbee's coupling to bound $\sum_{t \in A_k} \tilde{X}'_t \Delta_{(k)}$. By Theorem 16.2.1 of Athreya and Lahiri (2006), on an enlarged probability space, there exist random variables $\{\bar{X}_t\}_{t \in A_k}$ such that (1) $\{\bar{X}_t\}_{t \in A_k}$ and $\{\tilde{X}_t\}_{t \in A_k}$ have the same distribution, (2) $\{\bar{X}_t\}_{t \in A_k}$ is independent of data in $\{1, ..., T_0\} \backslash H_k$ and (3) $P(\{\bar{X}_t\}_{t \in A_k} \neq \{\tilde{X}_t\}_{t \in A_k}) \leq \beta_{\mathrm{mix}}(\gamma_T)$. Notice that $\Delta_{(k)}$ is independent of $\{\bar{X}_t\}_{t \in A_k}$. Hence,

$$E\left[\left(\sum_{t \in A_k} \bar{X}'_t \Delta_{(k)} \right)^2 \Big| \Delta_{(k)} \right] = \Delta'_{(k)} E\left[\left(\sum_{t \in A_k} \bar{X}_t \right) \left(\sum_{t \in A_k} \bar{X}_t \right)' \right] \Delta_{(k)}$$

$$\overset{(i)}{\leq} |A_k| \kappa_1 \|\Delta_{(k)}\|_2^2,$$

where (i) follows by Assumption 3 and the fact that $\{\bar{X}_t\}_{t \in A_k}$ and $\{\tilde{X}_t\}_{t \in A_k}$ have the same distribution. Thus, $\sum_{t \in A_k} \bar{X}'_t \Delta_{(k)} = O_P(\sqrt{|A_k|} \|\Delta_{(k)}\|_2)$. Since $P(\{\bar{X}_t\}_{t \in A_k} \neq \{\tilde{X}_t\}_{t \in A_k}) \leq \beta_{\mathrm{mix}}(\gamma_T) = o(1)$, it follows that

$$\sum_{t \in A_k} \tilde{X}'_t \Delta_{(k)} = O_P(\sqrt{|A_k|} \|\Delta_{(k)}\|_2). \tag{32}$$

Now by (31) and (32),

$$\sum_{t \in H_k} \tilde{X}'_t \Delta_{(k)} = O_P\left(\rho_T \gamma_T \|\Delta_{(k)}\|_1 + \sqrt{|A_k|} \|\Delta_{(k)}\|_2 \right)$$

$$\overset{(i)}{=} O_P\left(\rho_T \gamma_T \|\Delta_{(k)}\|_1 + \sqrt{r} \|\Delta_{(k)}\|_2 \right),$$

where (i) follows by the assumption that $\gamma_T = o(T_0)$ and $|A_k| < |H_k| = r$. Similarly, we can show

$$T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{X}'_t \Delta = O_P\left(T_1^{-1} \rho_T \gamma_T \|\Delta_{(k)}\|_1 + T_1^{-1/2} \|\Delta_{(k)}\|_2 \right).$$

Since $r \asymp \min\{T_0, T_1\}$, the above two displays and (30) imply

$$\left| \hat{\tau}_k - \tau - \left(\frac{1}{T_1} \sum_{t=T_0+1}^{T} u_t - \frac{1}{|H_k|} \sum_{t \in H_k} u_t \right) \right| = O_P\left(\frac{\rho_T \gamma_T \|\Delta_{(k)}\|_1}{\min\{T_0, T_1\}} + \frac{\|\Delta_{(k)}\|_2}{\sqrt{\min\{T_0, T_1\}}} \right).$$

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By Assumptions 2 and 3 as well as the covariance-stationarity of u_t, we have

$$\sqrt{\min\{T_0, T_1\}} \left| \hat{\tau}_k - \tau - \left(\frac{1}{T_1} \sum_{t=T_0+1}^{T} \tilde{u}_t - \frac{1}{|H_k|} \sum_{t \in H_k} \tilde{u}_t \right) \right| = o_P(1),$$

where $\tilde{u}_t = u_t - E(u_t)$. The desired result in Theorem 1 follows from Assumption 2 and the usual CLT for dependent processes, e.g., Theorem 5.20 of White (2001).

G.7 Proof of Theorem 2

Part (i) is a direct consequence of Theorem 1. For Part (ii), recall that

$$\mathbb{T}_K = \frac{\sqrt{K}(\hat{\tau} - \tau_0)}{\sqrt{1 + \frac{Kr}{T_1}} \times \sqrt{(K-1)^{-1} \sum_{k=1}^{K} (\hat{\tau}_{(k)} - \hat{\tau})^2}}.$$

We notice that $r/T_1 \to \min\{c_0/K, 1\}$. Then Theorem 1 and the continuous mapping theorem imply that $\mathbb{T}_K \xrightarrow{d} \mathcal{T}_K$, where

$$\mathcal{T}_K = \frac{\sqrt{K} \left(\sqrt{\min\{c_0, 1\}} \xi_0 - \sqrt{g_{c_0,K}} \bar{\xi} \right)}{\sqrt{1 + \min\{c_0, K\}} \times \sqrt{(K-1)^{-1} \sum_{k=1}^{K} g_{c_0,K} (\xi_k - \bar{\xi})^2}}$$

$$= \frac{\sqrt{\frac{K}{g_{c_0,K}(1+\min\{c_0,K\})}} \left(\sqrt{\min\{c_0, 1\}} \xi_0 - \sqrt{g_{c_0,K}} \bar{\xi} \right)}{\sqrt{(K-1)^{-1} \sum_{k=1}^{K} (\xi_k - \bar{\xi})^2}},$$

where $\bar{\xi} = K^{-1} \sum_{k=1}^{K} \xi_k$.

Notice that $\bar{\xi}$ is independent of $\sum_{k=1}^{K} (\xi_k - \bar{\xi})^2$. Therefore, the numerator is independent of the denominator. The variance of the numerator is

$$\frac{K}{g_{c_0,K}(1+\min\{c_0,K\})} \times \left(\min\{c_0, 1\} + g_{c_0,K} K^{-1} \right) = \begin{cases} 1 & \text{if } 1 \le c_0 \le K \\ 1 & \text{if } c_0 < 1 \\ 1 & \text{if } c_0 > K. \end{cases}$$

Hence, the numerator has a standard normal distribution. It follows that $\mathcal{T}_K \sim t_{K-1}$. The proof is complete. $\qquad\square$

G.8 Proof of Theorem 3

We start with an auxiliary lemma, which is an analog of Lemma 1.

Lemma 6. *Let the assumptions of Theorem 3 hold. Then, for $1 \leq k \leq K$,*

$$\hat{\tau}_k - \tau = \left(\frac{1}{T_1} \sum_{t=T_0+1}^{T} u_t - \frac{1}{|H_k|} \sum_{t \in H_k} u_t \right) + \left(\frac{1}{|H_k|} \sum_{t \in H_k} \tilde{Z}_t - \frac{1}{T_1} \sum_{t=T_0+1}^{T} \tilde{Z}_t \right)' \Delta_{(k)},$$

where $\tilde{Z}_t = Z_t - E(Z_t)$ and $\Delta_{(k)} = \hat{w}_{(k)} - w_*$.

Proof of Lemma 6. Let $\mu = E(Z_t)$. Notice that because $\mathbf{1}_N' w_* = \mathbf{1}_N' \hat{w}_{(k)} = 1$, for $T_0 + 1 \leq t \leq T$,

$$Y_t - X_t' \hat{w}_{(k)} = \alpha_t + u_t - Z_t' \Delta_{(k)} = \alpha_t + u_t - \mu' \Delta_{(k)} - \tilde{Z}_t' \Delta_{(k)}$$

and for $t \in H_k$

$$Y_t - X_t' \hat{w}_{(k)} = u_t - Z_t' \Delta_{(k)} = u_t - \mu' \Delta_{(k)} - \tilde{Z}_t' \Delta_{(k)}.$$

Therefore,

$$\hat{\tau}_k - \tau = T_1^{-1} \sum_{t=T_0+1}^{T} (Y_t - X_t' \hat{w}_{(k)}) - \frac{1}{|H_k|} \sum_{t \in H_k} (Y_t - X_t' \hat{w}_{(k)}) - \tau$$

$$= T_1^{-1} \sum_{t=T_0+1}^{T} u_t - \frac{1}{|H_k|} \sum_{t \in H_k} u_t + \frac{1}{|H_k|} \sum_{t \in H_k} \tilde{Z}_t' \Delta_{(k)} - T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{Z}_t' \Delta_{(k)}.$$

The proof is complete. $\qquad\square$

Proof of Theorem 3. By Equation (15), $\hat{w}_{(k)}$ only depends on $\{Z_t\}_{t \in H_{(-k)}}$ and, thus, is approximately independent of $\{Z_t\}_{t \in H_{(k)} \cup \{T_0+1,\dots,T\}}$. Therefore, using Lemma 6 and the same arguments as in Theorem 1, one can show that

$$\left| \hat{\tau}_k - \tau - \left(\frac{1}{T_1} \sum_{t=T_0+1}^{T} u_t - \frac{1}{|H_k|} \sum_{t \in H_k} u_t \right) \right| = O_P \left(\frac{\rho_T \gamma_T \|\Delta_{(k)}\|_1}{\min\{T_0, T_1\}} + \frac{\|\Delta_{(k)}\|_2}{\sqrt{\min\{T_0, T_1\}}} \right).$$

The result now follows by Assumption 2 and the same arguments as in Theorems 1 and 2 and Corollary 1. $\qquad\square$

G.9 Proof of Theorem 4

We first derive a result on the error bounds for the SC estimator. Then we prove Theorem 4.

Lemma 7. *Assume that $Y = Xw + u$ and $X = \theta \mathbf{1}_N' + Z + \xi \beta'$, where $u \in \mathbb{R}^T$, $w \in \mathbb{R}^N$ satisfies $\mathbf{1}_N' w = 1$ and $w \geq 0$. Consider the estimator*

$$\hat{w} = \arg \min_{v \in \mathbb{R}^N} \|Y - Xv\|_2^2 \qquad s.t. \qquad v \geq 0, \ \mathbf{1}_N' v = 1.$$

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Suppose that the eigenvalues of $EZ'Z/T$ are between c_0^{-1} and c_0. Then on the event $\mathcal{M}$,

$$\|\hat{w} - w\|_2 \leq \sqrt{\frac{c_0}{T} \left[8\lambda_1^2 + 2c_1^2 + 4\lambda_2 + 4\lambda_3\right]}$$

and

$$|\beta'(\hat{w} - w)| \leq \|\xi\|_2^{-1} \left[6\lambda_1 + 3c_1 + 4\sqrt{\lambda_2}\right],$$

where

$$\mathcal{M} = \{\|Z'\xi\|_\infty \leq \lambda_1 \|\xi\|_2\} \bigcap \{|u'\xi| \leq c_1 \|\xi\|_2\} \bigcap \{\|Z'u\|_\infty \leq \lambda_2\} \bigcap \{\|Z'Z - EZ'Z\|_\infty \leq \lambda_3\}.$$

Proof of Lemma 7. Let $\Delta = \hat{w} - w$ and $\Sigma_Z = EZ'Z/T$. We proceed in two steps.

Step 1: bound $\|\Delta\|_2$.

Recall that $Y = Xw + u$. By definition, we have $\|Y - X\hat{w}\|_2^2 \leq \|Y - Xw\|_2^2$. This means that

$$\|X\Delta\|_2^2 \leq 2u'X\Delta.$$

We notice that $\mathbf{1}_N'\Delta = 0$, $X = \theta\mathbf{1}_N' + Z + \xi\beta'$ and thus

$$X\Delta = Z\Delta + \xi\beta'\Delta.$$

Therefore, we have

$$\|Z\Delta\|_2^2 + \|\xi\|_2^2(\beta'\Delta)^2 + 2\Delta'Z'\xi\beta'\Delta \leq 2u'Z\Delta + 2u'\xi\beta'\Delta.$$

We then have

$$\left[\|\xi\|_2(\beta'\Delta) + \|\xi\|_2^{-1}(\Delta'Z'\xi - u'\xi)\right]^2 \leq \|\xi\|_2^{-2}(\Delta'Z'\xi - u'\xi)^2 + 2u'Z\Delta - \|Z\Delta\|_2^2. \quad (33)$$

It follows that

$$\|\xi\|_2^{-2}(\Delta'Z'\xi - u'\xi)^2 + 2u'Z\Delta - \|Z\Delta\|_2^2 \geq 0.$$

Notice that $\|\Delta\|_1 \leq \|\hat{w}\|_1 + \|w\|_1 = 2$. Hence,

$$(\Delta'Z'\xi - u'\xi)^2 \leq 2(\Delta'Z'\xi)^2 + 2(u'\xi)^2 \leq 2\|\Delta\|_1^2\|Z'\xi\|_\infty^2 + 2c_1^2\|\xi\|_2^2 \leq (8\lambda_1^2 + 2c_1^2)\|\xi\|_2^2. \quad (34)$$

By the above two displays, we have

$$8\lambda_1^2 + 2c_1^2 + 2u'Z\Delta \geq \|Z\Delta\|_2^2.$$

Moreover, we notice that $|u'Z\Delta| \leq \lambda_2\|\Delta\|_1$ and

$$\|Z\Delta\|_2^2 = \Delta'EZ'Z\Delta + \Delta'(Z'Z - EZ'Z)\Delta$$

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$$\geq \Delta' E Z' Z \Delta - \|Z'Z - EZ'Z\|_\infty \|\Delta\|_1^2.$$

By $\|\Delta\|_1 \leq 2$, we have that

$$8\lambda_1^2 + 2c_1^2 + 4\lambda_2 \geq T\Delta'\Sigma_Z\Delta - 4\lambda_3.$$

Therefore,

$$\Delta'\Sigma_Z\Delta \leq \frac{1}{T}\left[8\lambda_1^2 + 2c_1^2 + 4\lambda_2 + 4\lambda_3\right]. \tag{35}$$

We obtain the bound on $\|\Delta\|_2$ by $\lambda_{\min}(\Sigma_Z) \geq c_0^{-1}$.

Step 2: bound $\beta'\Delta$.

By (33) and (34), we have

$$\left[\|\xi\|_2(\beta'\Delta) + \|\xi\|_2^{-1}(\Delta'Z'\xi - u'\xi)\right]^2 \leq \|\xi\|_2^{-2}(\Delta'Z'\xi - u'\xi)^2 + 2u'Z\Delta$$

$$\leq 8\lambda_1^2 + 2c_1^2 + 2u'Z\Delta \leq 8\lambda_1^2 + 2c_1^2 + 2\|Z'u\|_\infty\|\Delta\|_1 \leq 8\lambda_1^2 + 2c_1^2 + 4\lambda_2.$$

Then

$$\|\xi\|_2 \cdot |\beta'\Delta| \leq \|\xi\|_2^{-1} \cdot |\Delta'Z'\xi - u'\xi| + \sqrt{8\lambda_1^2 + 2c_1^2 + 4\lambda_2}$$

$$\overset{(i)}{\leq} \sqrt{8\lambda_1^2 + 2c_1^2} + \sqrt{8\lambda_1^2 + 2c_1^2 + 4\lambda_2}$$

$$\leq 2\sqrt{8\lambda_1^2 + 2c_1^2 + 4\lambda_2} \leq 6\lambda_1 + 3c_1 + 4\sqrt{\lambda_2}.$$

where (i) follows by (34). The proof is complete. $\qquad\square$

Proof of Theorem 4. Let $\Delta_{(k)} = \hat{w}_{(k)} - w$. Notice that

$$\hat{\tau}_k - \tau = T_1^{-1} \sum_{t=T_0+1}^{T} (Y_t - X_t'\hat{w}_{(k)}) - \frac{1}{|H_k|} \sum_{t\in H_k} (Y_t - X_t'\hat{w}_{(k)}) - \tau$$

$$= T_1^{-1} \sum_{t=T_0+1}^{T} (\alpha_t + u_t - X_t'\Delta_{(k)}) - \frac{1}{|H_k|} \sum_{t\in H_k} (u_t - X_t'\Delta_{(k)}) - \tau$$

$$= T_1^{-1} \sum_{t=T_0+1}^{T} (u_t - X_t'\Delta_{(k)}) - \frac{1}{|H_k|} \sum_{t\in H_k} (u_t - X_t'\Delta_{(k)})$$

$$= T_1^{-1} \sum_{t=T_0+1}^{T} u_t - \frac{1}{|H_k|} \sum_{t\in H_k} u_t + \frac{1}{|H_k|} \sum_{t\in H_k} X_t'\Delta_{(k)} - T_1^{-1} \sum_{t=T_0+1}^{T} X_t'\Delta_{(k)}. \tag{36}$$

We notice that

$$\sum_{t\in H_k} X_t'\Delta_{(k)} = \sum_{t\in H_k} Z_t'\Delta_{(k)} + \left(\sum_{t\in H_k} \xi_t\right)(\beta'\Delta_{(k)}).$$

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Since $\Delta_{(k)}$ depends on $\{(\theta_t, Z_t, \xi_t)\}_{t\in H_{(-k)}}$ and $\{Z_t\}_{t\in H_{(-k)}}$ is independent of $\{(\theta_t, \xi_t)\}_{t=1}^T$, it follows that conditional on $\{(\theta_t, \xi_t)\}_{t=1}^T$, we can still use the coupling argument in the proof of Theorem 1 and obtain

$$\sum_{t\in H_k} Z_t'\Delta_{(k)} = O_P\left(\rho_T\gamma_T + \sqrt{|H_k|}\cdot\|\Delta_{(k)}\|_2\right).$$

By Assumption 7, we apply Lemma 7 and obtain

$$\|\Delta_{(k)}\|_2 = O_P\left((T^{-1}\log N)^{1/4}\right) = o_P(1) \tag{37}$$

and

$$|\beta'\Delta_{(k)}| = O_P\left(\frac{(\log N)^{1/2}}{\sqrt{\sum_{t\in H_{(-k)}}\xi_t^2}}\right).$$

Therefore,

$$\frac{\sqrt{\min\{T_0, T_1\}}}{|H_k|}\sum_{t\in H_k} X_t'\Delta_{(k)}$$

$$= \frac{\sqrt{\min\{T_0, T_1\}}}{|H_k|}\sum_{t\in H_k} Z_t'\Delta_{(k)} + \frac{\sqrt{\min\{T_0, T_1\}}}{|H_k|}\left(\sum_{t\in H_k}\xi_t\right)(\beta'\Delta_{(k)})$$

$$= \frac{\sqrt{\min\{T_0, T_1\}}}{|H_k|}O_P\left(\rho_T\gamma_T + \sqrt{|H_k|}\cdot\|\Delta_{(k)}\|_2\right)$$

$$+ \frac{\sqrt{\min\{T_0, T_1\}}}{|H_k|}O_P\left(\left(\sum_{t\in H_k}\xi_t\right)\cdot\frac{(\log N)^{1/2}}{\sqrt{\sum_{t\in H_{(-k)}}\xi_t^2}}\right)$$

$$\stackrel{(i)}{=} o_P(1) + T_1^{-1/2}O_P\left((\log N)^{1/2}\frac{\sum_{t\in H_k}\xi_t}{\sqrt{\sum_{t\in H_{(-k)}}\xi_t^2}}\right)$$

$$\stackrel{(ii)}{=} o_P(1) + T_1^{-1/2}O_P\left((\log N)^{1/2}\cdot T_1 T^{-1/2}\right) \stackrel{(iii)}{=} o_P(1), \tag{38}$$

where (i) follows by (37), $\min\{T_1, T_0\} \asymp T_1$, $|H_k| \asymp T_1$, $T_0 \asymp T$, $T_1^{-1/2}\rho_T\gamma_T = o(1)$ (Assumption 3), (ii) follows by Assumption 6, and (iii) follows by $T_1 \ll T_0/\log N$.

Similarly, we can show that

$$\sum_{t=T_0+1}^T Z_t'\Delta_{(k)} = O_P\left(\rho_T\gamma_T + \sqrt{T_1}\|\Delta_{(k)}\|_2\right)$$

and thus

$$\frac{\sqrt{\min\{T_0, T_1\}}}{T_1}\sum_{t=T_0+1}^T X_t'\Delta_{(k)}$$

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$$
\begin{aligned}
&= \frac{\sqrt{\min\{T_0, T_1\}}}{T_1} \sum_{t=T_0+1}^{T} Z_t' \Delta_{(k)} + \frac{\sqrt{\min\{T_0, T_1\}}}{T_1} \left(\sum_{t=T_0+1}^{T} \xi_t \right) (\beta' \Delta_{(k)}) \\
&= \frac{\sqrt{\min\{T_0, T_1\}}}{T_1} O_P \left(\rho_T \gamma_T + \sqrt{T_1} \| \Delta_{(k)} \|_2 \right) \\
&\quad + \frac{\sqrt{\min\{T_0, T_1\}}}{T_1} O_P \left(\left(\sum_{t=T_0+1}^{T} \xi_t \right) \frac{(\log N)^{1/2}}{\sqrt{\sum_{t \in H_{(-k)}} \xi_t^2}} \right) \\
&\overset{\text{(i)}}{=} o_P(1) + T_1^{-1/2} O_P \left((\log N)^{1/2} \frac{\sum_{t=T_0+1}^{T} \xi_t}{\sqrt{\sum_{t \in H_{(-k)}} \xi_t^2}} \right) \\
&\overset{\text{(ii)}}{=} o_P(1) + T_1^{-1/2} O_P \left((\log N)^{1/2} \cdot T_1 T^{-1/2} \right) \overset{\text{(iii)}}{=} o_P(1), \tag{39}
\end{aligned}
$$

where (i) follows by (37), $\min\{T_1, T_0\} \asymp T_1$ and $T_1^{-1/2} \rho_T \gamma_T = o(1)$, (ii) follows by Assumption 6 and (iii) follows by $T_1 \ll T_0 / \log N$.

Now we combine (36) with (38) and (39), obtaining

$$
\begin{aligned}
&\sqrt{\min\{T_0, T_1\}} (\hat{\tau}_k - \tau) \\
&= \sqrt{\frac{\min\{T_0, T_1\}}{T_1}} \left(T_1^{-1/2} \sum_{t=T_0+1}^{T} u_t \right) - \sqrt{\frac{\min\{T_0, T_1\}}{|H_k|}} \left(|H_k|^{-1/2} \sum_{t \in H_k} u_t \right) \\
&\quad + \frac{\sqrt{\min\{T_0, T_1\}}}{|H_k|} \sum_{t \in H_k} X_t' \Delta_{(k)} - \frac{\sqrt{\min\{T_0, T_1\}}}{T_1} \sum_{t=T_0+1}^{T} X_t' \Delta_{(k)} \\
&= \sqrt{\frac{\min\{T_0, T_1\}}{T_1}} \left(T_1^{-1/2} \sum_{t=T_0+1}^{T} u_t \right) - \sqrt{\frac{\min\{T_0, T_1\}}{|H_k|}} \left(|H_k|^{-1/2} \sum_{t \in H_k} u_t \right) + o_P(1).
\end{aligned}
$$

Since $T_1 \ll T_0$ and $|H_k| = \min\{T_0/K, T_1\}$, we have $\min\{T_0, T_1\}/T_1 \to 1$ and $\min\{T_0, T_1\}/|H_k| \to 1$. The first claim of the theorem follows.

For the second claim, we notice that $(T_1^{-1/2} \sum_{t=T_0+1}^{T} u_t, |H_1|^{-1/2} \sum_{t \in H_1} u_t, ..., |H_K|^{-1/2} \sum_{t \in H_k} u_t)'$ converges in distribution to the normal distribution with mean zero and variance matrix $I_{K+1} \sigma^2$, where σ^2 is the long-run variance of u_t. Notice that

$$
\mathbb{T}_K = \frac{\sqrt{\frac{KT_1}{|H_1|K + T_1}} \cdot (\hat{\tau} - \tau)}{\sqrt{\frac{1}{K-1} \sum_{k=1}^{K} (\hat{\tau}_{(k)} - \hat{\tau})^2}}.
$$

By the same argument as in the proof of Theorem 2, asymptotically the numerator has $N(0,1)$ distribution and is independent of the denominator, which asymptotically has a χ^2_{K-1} distribution divided by $K-1$. Therefore, $\mathbb{T}_K$ converges in distribution to the student t-distribution with $K-1$ degrees of freedom. The proof is complete. $\square$

G.10 Proof of Theorem 5

Before we prove Theorem 5, we introduce the some notations and establish a preliminary result. For any $i \in \{0, ..., N\}$ and any $w \in \mathcal{W}^{SC}$, we have that

$$Y_{it}(0) - Y_{-i,t}(0)'w = \theta_t + V_{it} - (\theta_t \mathbf{1}'_N + V'_{-i,t})w = V_{it} - V'_{-i,t}w.$$

Due to the covariance-stationarity of V_t, the distribution of the above quantity does not depend on t. Hence, we can define the following time-independent quantity:

$$w^{(i)}_* \in \underset{w \in \mathcal{W}^{SC}}{\arg\min} \, E(Y_{it}(0) - Y'_{-i,t}(0)w)^2 = \underset{w \in \mathcal{W}^{SC}}{\arg\min} \, E(V_{it} - V'_{-i,t}w)^2.$$

To state the next result, let $\Delta^{(i)}_{(k)} = \hat{w}^{(i)}_{(k)} - w^{(i)}_*$.

Lemma 8. *Let Assumptions 8 and 9 hold. Then* $\max_{1 \le k \le K} \max_{0 \le i \le N} \|\Delta^{(i)}_{(k)}\|_2 = o_P(1)$.

Proof of Lemma 8. Let $f_i(w)$ be the $(N+1)$-dimensional vector whose i-th entry is 1 and whose other entries are $-w$. Then

$$\underset{w \in \mathcal{W}^{SC}}{\arg\min} \, E(V_{it} - V'_{-i,t}w)^2 = \underset{w \in \mathcal{W}^{SC}}{\arg\min} \, E(V'_t f_i(w))^2 = \underset{w \in \mathcal{W}^{SC}}{\arg\min} \, f_i(w)'\Sigma f_i(w),$$

where $\Sigma = EV_t V'_t$. Similarly,

$$\underset{w \in \mathcal{W}^{SC}}{\arg\min} \sum_{t \in H_{(-k)}} (Y_{it} - Y'_{-i,t}w)^2 = \underset{w \in \mathcal{W}^{SC}}{\arg\min} \sum_{t \in H_{(-k)}} (V_{it} - V'_{-i,t}w)^2$$

$$= \underset{w \in \mathcal{W}^{SC}}{\arg\min} \sum_{t \in H_{(-k)}} (V'_t f_i(w))^2 = \underset{w \in \mathcal{W}^{SC}}{\arg\min} \, f_i(w)'\hat{\Sigma}_{(-k)} f_i(w),$$

where $\hat{\Sigma}_{(-k)} = |H_{(-k)}|^{-1} \sum_{t \in H_{(-k)}} V_t V'_t$. Let $\Delta^{(i)}_{(k)} = f_i(\hat{w}^{(i)}_{(k)}) - f_i(w^{(i)}_*)$. By construction,

$$[f_i(w^{(i)}_*) + \Delta^{(i)}_{(k)}]'\hat{\Sigma}_{(-k)}[f_i(w^{(i)}_*) + \Delta^{(i)}_{(k)}] \le f_i(w^{(i)}_*)'\hat{\Sigma}_{(-k)} f_i(w^{(i)}_*).$$

Thus,
$$(\Delta^{(i)}_{(k)})'\hat{\Sigma}_{(-k)}\Delta^{(i)}_{(k)} \le -2f_i(w^{(i)}_*)'\hat{\Sigma}_{(-k)}\Delta^{(i)}_{(k)}.$$

Similar to the argument of (26) in the proof of Lemma 2, we have that

$$f_i(w^{(i)}_*)'\Sigma\Delta^{(i)}_{(k)} \ge 0.$$

The above two displays imply

$$(\Delta^{(i)}_{(k)})'\Sigma\Delta^{(i)}_{(k)} - \|\hat{\Sigma}_{(-k)} - \Sigma\|_\infty \cdot \|\Delta^{(i)}_{(k)}\|_1^2 \le (\Delta^{(i)}_{(k)})'\hat{\Sigma}_{(-k)}\Delta^{(i)}_{(k)}$$

$$\leq -2f_i(w_*^{(i)})'\hat{\Sigma}_{(-k)}\Delta_{(k)}^{(i)} \leq 2\|f_i(w_*^{(i)})\|_1\|\hat{\Sigma}_{(-k)} - \Sigma\|_\infty\|\Delta_{(k)}^{(i)}\|_1.$$

By construction, $\|f_i(w_*^{(i)})\|_1 = 2$ and $\|\Delta_{(k)}^{(i)}\|_1 = \|\hat{w}_{(k)}^{(i)} - w_*^{(i)}\|_1 \leq 2$. It follows that $(\Delta_{(k)}^{(i)})'\Sigma\Delta_{(k)}^{(i)} \leq 12\|\hat{\Sigma}_{(-k)} - \Sigma\|_\infty$. Since the eigenvalues of Σ are bounded below by $1/\kappa_1$, this means that

$$\max_{0\leq i\leq N, 0\leq k\leq K} \|\Delta_{(k)}^{(i)}\|_2^2 \leq 12\kappa_1 \max_{0\leq k\leq K} \|\hat{\Sigma}_{(-k)} - \Sigma\|_\infty.$$

The result follows by the assumption of $\max_{0\leq k\leq K} \|\hat{\Sigma}_{(-k)} - \Sigma\|_\infty = o_P(1)$. $\square$

Proof of Theorem 5. Define $\tilde{V}_{it} = V_{it} - EV_{it}$ and $u_{it} = \tilde{V}_{it} - \tilde{V}_{-i,t}'w_*^{(i)}$. Since $\hat{w}_{(k)}^{(i)} \in \mathcal{W}^{SC}$, we have that $Y_{it}(0) - Y_{-i,t}(0)'\hat{w}_{(k)}^{(i)} = V_{it} - V_{-i,t}'\hat{w}_{(k)}^{(i)}$. Hence,

$$T_1^{-1}\sum_{i=0}^{N}\sum_{t=T_0+1}^{T}(Y_{it} - Y_{-i,t}'\hat{w}_{(k)}^{(i)})\mathbf{1}\{D_i = 1\}$$

$$= T_1^{-1}\sum_{i=0}^{N}\sum_{t=T_0+1}^{T}(\alpha_{it} + Y_{it}(0) - Y_{-i,t}(0)'\hat{w}_{(k)}^{(i)})\mathbf{1}\{D_i = 1\}$$

$$= \tau + T_1^{-1}\sum_{i=0}^{N}\sum_{t=T_0+1}^{T}(Y_{it}(0) - Y_{-i,t}(0)'\hat{w}_{(k)}^{(i)})\mathbf{1}\{D_i = 1\}$$

$$= \tau + T_1^{-1}\sum_{i=0}^{N}\sum_{t=T_0+1}^{T}(V_{it} - V_{-i,t}'\hat{w}_{(k)}^{(i)})\mathbf{1}\{D_i = 1\}$$

and similarly,

$$\frac{1}{|H_k|}\sum_{i=0}^{N}\sum_{t\in H_k}(Y_{it} - Y_{-i,t}'\hat{w}_{(k)}^{(i)})\mathbf{1}\{D_i = 1\} = \frac{1}{|H_k|}\sum_{i=0}^{N}\sum_{t\in H_k}(V_{it} - V_{-i,t}'\hat{w}_{(k)}^{(i)})\mathbf{1}\{D_i = 1\}.$$

Therefore, letting $\Delta_{(k)}^{(i)} = \hat{w}_{(k)}^{(i)} - w_*^{(i)}$, we have

$$\hat{\tau}_k - \tau = \sum_{i=0}^{N}\left(T_1^{-1}\sum_{t=T_0+1}^{T}(V_{it} - V_{-i,t}'\hat{w}_{(k)}^{(i)}) - \frac{1}{|H_k|}\sum_{t\in H_k}(V_{it} - V_{-i,t}'\hat{w}_{(k)}^{(i)})\right)\cdot\mathbf{1}\{D = i\}$$

$$= \sum_{i=0}^{N}\left(T_1^{-1}\sum_{t=T_0+1}^{T}(\tilde{V}_{it} - \tilde{V}_{-i,t}'\hat{w}_{(k)}^{(i)}) - \frac{1}{|H_k|}\sum_{t\in H_k}(\tilde{V}_{it} - \tilde{V}_{-i,t}'\hat{w}_{(k)}^{(i)})\right)\cdot\mathbf{1}\{D = i\}$$

$$+ \sum_{i=0}^{N}\left(T_1^{-1}\sum_{t=T_0+1}^{T}(EV_{it} - (EV_{-i,t})'\hat{w}_{(k)}^{(i)}) - \frac{1}{|H_k|}\sum_{t\in H_k}(EV_{it} - (EV_{-i,t})'\hat{w}_{(k)}^{(i)})\right)\cdot\mathbf{1}\{D = i\}$$

$$\overset{(i)}{=} \sum_{i=0}^{N}\left(T_1^{-1}\sum_{t=T_0+1}^{T}(\tilde{V}_{it} - \tilde{V}_{-i,t}'\hat{w}_{(k)}^{(i)}) - \frac{1}{|H_k|}\sum_{t\in H_k}(\tilde{V}_{it} - \tilde{V}_{-i,t}'\hat{w}_{(k)}^{(i)})\right)\cdot\mathbf{1}\{D = i\}$$

$$= \sum_{i=0}^{N} \left(T_1^{-1} \sum_{t=T_0+1}^{T} u_{it} - \frac{1}{|H_k|} \sum_{t\in H_k} u_{it} \right) \cdot \mathbf{1}\{D=i\}$$

$$+ \sum_{i=0}^{N} \left(\frac{1}{|H_k|} \sum_{t\in H_k} \tilde{V}_{-i,t} - T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{V}_{-i,t} \right)' \Delta_{(k)}^{(i)} \cdot \mathbf{1}\{D=i\}, \tag{40}$$

where (i) follows by the covariance-stationarity of V_t.

For any $i \in \{0, ..., N\}$, let $\bar{\Delta}_{(k)}^{(i)}$ be the $(N+1)$-dimensional vector whose i-th component is zero and other components correspond to $\Delta_{(k)}^{(i)}$. Then $\tilde{V}_{-i,t}'\Delta_{(k)}^{(i)} = \tilde{V}_t'\bar{\Delta}_{(k)}^{(i)}$, $\|\bar{\Delta}_{(k)}^{(i)}\|_2^2 = \|\Delta_{(k)}^{(i)}\|_2^2$ and $\|\bar{\Delta}_{(k)}^{(i)}\|_1 = \|\Delta_{(k)}^{(i)}\|_1 \leq 2$. Hence,

$$\sum_{i=0}^{N} \left(\frac{1}{|H_k|} \sum_{t\in H_k} \tilde{V}_{-i,t} - T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{V}_{-i,t} \right)' \Delta_{(k)}^{(i)} \cdot \mathbf{1}\{D=i\}$$

$$= \frac{1}{|H_k|} \sum_{t\in H_k} \sum_{i=0}^{N} \tilde{V}_{-i,t}'\Delta_{(k)}^{(i)} \cdot \mathbf{1}\{D=i\} - T_1^{-1} \sum_{t=T_0+1}^{T} \sum_{i=0}^{N} \tilde{V}_{-i,t}'\Delta_{(k)}^{(i)} \cdot \mathbf{1}\{D=i\}$$

$$= \frac{1}{|H_k|} \sum_{t\in H_k} \sum_{i=0}^{N} \tilde{V}_t'\bar{\Delta}_{(k)}^{(i)} \cdot \mathbf{1}\{D=i\} - T_1^{-1} \sum_{t=T_0+1}^{T} \sum_{i=0}^{N} \tilde{V}_t'\bar{\Delta}_{(k)}^{(i)} \cdot \mathbf{1}\{D=i\}$$

$$= \frac{1}{|H_k|} \sum_{t\in H_k} \tilde{V}_t'\Delta_{(k)} - T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{V}_t'\Delta_{(k)},$$

where $\Delta_{(k)} = \sum_{i=0}^{N} \bar{\Delta}_{(k)}^{(i)} \cdot \mathbf{1}\{D=i\}$. By the same coupling argument as in the proof of Theorem 1, we have that

$$\left| \frac{1}{|H_k|} \sum_{t\in H_k} \tilde{V}_t'\Delta_{(k)} - T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{V}_t'\Delta_{(k)} \right| = O_P \left(\frac{\rho_T\gamma_T\|\Delta_{(k)}\|_1}{\min\{T_0,T_1\}} + \frac{\|\Delta_{(k)}\|_2}{\sqrt{\min\{T_0,T_1\}}} \right),$$

where $\rho_T\gamma_T \to 0$. Notice that $\|\Delta_{(k)}\|_1 = \sum_{i=0}^{N} \|\bar{\Delta}_{(k)}^{(i)}\|_1 \cdot \mathbf{1}\{D=i\} \leq 2$ and $\|\Delta_{(k)}\|_2 = \sum_{i=0}^{N} \|\bar{\Delta}_{(k)}^{(i)}\|_2 \cdot \mathbf{1}\{D=i\} \leq \max_{0\leq i\leq N} \|\bar{\Delta}_{(k)}^{(i)}\|_2 = \max_{0\leq i\leq N} \|\Delta_{(k)}^{(i)}\|_2 = o_P(1)$ due to Lemma 8. Hence,

$$\left| \frac{1}{|H_k|} \sum_{t\in H_k} \tilde{V}_t'\Delta_{(k)} - T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{V}_t'\Delta_{(k)} \right| = o_P \left(\frac{1}{\sqrt{\min\{T_0,T_1\}}} \right).$$

Let $\bar{w}_*^{(i)}$ be the $(N+1)$-dimensional vector whose i-th entry is 1 and the other entries correspond to $-w_*^{(i)}$. Then $u_{it} = \tilde{V}_{it} - \tilde{V}_{-i,t}'w_*^{(i)} = \tilde{V}_t'\bar{w}_*^{(i)}$. Therefore,

$$\sum_{i=0}^{N} \left(T_1^{-1} \sum_{t=T_0+1}^{T} u_{it} - \frac{1}{|H_k|} \sum_{t\in H_k} u_{it} \right) \cdot \mathbf{1}\{D=i\}$$

$$= \sum_{i=0}^{N} \left(T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{V}_t'\bar{w}_*^{(i)} - \frac{1}{|H_k|} \sum_{t\in H_k} \tilde{V}_t'\bar{w}_*^{(i)} \right) \cdot \mathbf{1}\{D=i\}$$

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$$= T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{V}_t' \bar{w}_* - \frac{1}{|H_k|} \sum_{t \in H_k} \tilde{V}_t' \bar{w}_*,$$

where $\bar{w}_* = \sum_{i=0}^{N} \bar{w}_*^{(i)} \mathbf{1}\{D = i\}$. The above two displays and (40) imply that

$$\sqrt{\min\{T_0, T_1\}} \left| \hat{\tau}_{(k)} - \tau - \left(T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{V}_t' \bar{w}_* - \frac{1}{|H_k|} \sum_{t \in H_k} \tilde{V}_t' \bar{w}_* \right) \right| = o_P(1).$$

Let $A_1 = \{1, ..., T\} \backslash \{T_0 - 2B_T + 1, ..., T_0 + 2B_T\}$ and $A_2 = \{T_0 - B_T, ..., T_0 + B_T\}$. By Berbee's coupling (e.g., Theorem 16.2.1 of Athreya and Lahiri (2006)), on an enlarged probability space, there exist random variables $\{\bar{V}_t\}_{t \in A_1}$ such that (1) $\{\bar{V}_t\}_{t \in A_1}$ and $\{\tilde{V}_t\}_{t \in A_1}$ have the same distribution, (2) $\{\bar{V}_t\}_{t \in A_1}$ is independent of $\{\tilde{V}_t\}_{t \in A_2}$ and (3) $P(\{\bar{V}_t\}_{t \in A_1} \neq \{\tilde{V}_t\}_{t \in A_1}) \le \beta_{\text{mix}}(B_T)$. Define $R_{T,k} = T_1^{-1} \sum_{t=T_0+2B_T+1}^{T} \tilde{V}_t' \bar{w}_* - \frac{1}{|H_k|} \sum_{t \in H_k \backslash A_1} \tilde{V}_t' \bar{w}_*$. Notice that

$$T_1^{-1} \sum_{t=T_0+1}^{T} \tilde{V}_t' \bar{w}_* - \frac{1}{|H_k|} \sum_{t \in H_k} \tilde{V}_t' \bar{w}_* - R_{T,k}$$

$$= T_1^{-1} \sum_{t=T_0+1}^{T_0+2B_T} \tilde{V}_t' \bar{w}_* - \frac{1}{|H_k|} \sum_{t \in H_k \bigcap A_1} \tilde{V}_t' \bar{w}_*$$

$$= O_P \left(\max_{1 \le t \le T} \|\tilde{V}_t\|_\infty \left(T_1^{-1} \cdot 2B_T + |H_k|^{-1} \cdot |H_k \bigcap A_1| \right) \cdot \|\bar{w}_*\|_1 \right)$$

$$\overset{(i)}{=} O_P \left(\rho_T \left(T_1^{-1} \cdot 2B_T + |H_k|^{-1} \cdot |H_k \bigcap A_1| \right) \right) \overset{(ii)}{=} o_P \left(\frac{1}{\sqrt{\min\{T_0, T_1\}}} \right),$$

where (i) follows by $\|\bar{w}_*\|_1 = \sum_{i=0}^{N} \|\bar{w}_*^{(i)}\|_1 \cdot \mathbf{1}\{D = i\} \le 1$ and (ii) follows by $|H_k \bigcap A_1| \le 2B_T$ and the assumption of $B_T^2 \ll \rho_T^{-2} \min\{T_0, T_1\}$.

By assumption, $\bar{w}_*$ is independent of $\{\tilde{V}_t\}_{t \in A_1}$ conditional on $\{\tilde{V}_t\}_{t \in A_2} \bigcup \{\theta_t\}_{t \in A_2}$. Thus, $\bar{w}_*$ is independent of $\{\bar{V}_t\}_{t \in A_1}$ conditional on $\{\tilde{V}_t\}_{t \in A_2} \bigcup \{\theta_t\}_{t \in A_2}$. Thus, conditional on $\{\bar{V}_t\}_{t \in A_2} \bigcup \{\theta_t\}_{t \in A_2}$, there exists σ^2 representing the long-run variance of $T^{-1} \sum_{t=1}^{T} \bar{V}_t' \bar{w}_*$ such that

$$\sqrt{\min\{T_0, T_1\}} \begin{pmatrix} T_1^{-1} \sum_{t=T_0+2B_T+1}^{T} \bar{V}_t' \bar{w}_* \\ \frac{1}{|H_1|} \sum_{t \in H_1 \backslash A_1} \bar{V}_t' \bar{w}_* \\ \frac{1}{|H_2|} \sum_{t \in H_2 \backslash A_1} \bar{V}_t' \bar{w}_* \\ \vdots \\ \frac{1}{|H_K|} \sum_{t \in H_K \backslash A_1} \bar{V}_t' \bar{w}_* \end{pmatrix} / \sigma \overset{d}{\to} \begin{pmatrix} \sqrt{\min\{c_0, 1\}} \xi_0 \\ \sqrt{g_{c_0,K}} \xi_1 \\ \sqrt{g_{c_0,K}} \xi_2 \\ \vdots \\ \sqrt{g_{c_0,K}} \xi_K \end{pmatrix},$$

where $\xi_0, ..., \xi_K$ are independent $N(0, 1)$ random variables. Since $R_{T,k} = T_1^{-1} \sum_{t=T_0+2B_T+1}^{T} \tilde{V}_t' \bar{w}_* - \frac{1}{|H_k|} \sum_{t \in H_k \backslash A_1} \tilde{V}_t' \bar{w}_*$ with probability at least $1 - \beta_{\text{min}}(B_T)$ and $B_T \to \infty$, the above three

displays imply

$$\sqrt{\min\{T_0, T_1\}} \begin{pmatrix} \hat{\tau}_{(1)} - \tau \\ \vdots \\ \hat{\tau}_{(K)} - \tau \end{pmatrix} / \sigma \xrightarrow{d} \begin{pmatrix} \sqrt{\min\{c_0, 1\}}\xi_0 - \sqrt{g_{c_0, K}}\xi_1 \\ \vdots \\ \sqrt{\min\{c_0, 1\}}\xi_0 - \sqrt{g_{c_0, K}}\xi_K \end{pmatrix}.$$

The desired result follows by the same argument as in Theorem 2 and Corollary 1. □

G.11 Proof of Theorem 6

Using similar arguments as in Lemma 1 and Theorem 1, we obtain

$$\tilde{\tau}_k - \tau_e = \frac{1}{|\tilde{H}_k|} \sum_{t \in \tilde{H}_k} (u_t + \tilde{\alpha}_t) - \frac{1}{|H_k|} \sum_{t \in H_k} u_t + \frac{1}{\sqrt{\min\{T_0, T_1\}}} \epsilon_{T,k},$$

where $\max_{1 \leq k \leq K} |\epsilon_{T,k}| = o_P(1)$, and, since $\tilde{\tau}_1, \ldots, \tilde{\tau}_K$ are obtained based on non-overlapping blocks of data,

$$\sqrt{\min\{T_0, T_1\}} \begin{pmatrix} \hat{\tau}_1 - \tau_e \\ \vdots \\ \hat{\tau}_K - \tau_e \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_K \end{pmatrix},$$

where $\xi_1, \ldots, \xi_K$ are independent and identically distributed mean zero normal variables. The result (i) now follows from classical arguments, and (ii) follows from the same arguments as in Corollary 1.

G.12 Proof of Theorem 7

We define the mapping $\psi : \mathbb{R}^K \to \mathbb{R}$ by

$$\psi(v_1, ..., v_K) = \frac{\sqrt{K}\bar{v}}{\sqrt{(K-1)^{-1} \sum_{k=1}^K (v_k - \bar{v})^2}} \quad \text{with} \quad \bar{v} = K^{-1} \sum_{j=1}^K v_j.$$

We define $\Sigma_{T,K}$ to be the covariance matrix of $\sqrt{G}(\hat{\beta}_1, ..., \hat{\beta}_K)$. We also define the matrix Σ_K to be the diagonal matrix whose diagonal entries are all equal to $TE(\hat{\beta}_1^2)$. Finally, we define $\xi \sim N(0, I_K)$. Hence, $\mathbf{T}_K$ has the same distribution as $\psi(\Sigma_{T,K}^{1/2}\xi)$. Moreover, $\psi(\Sigma_K^{1/2}\xi)$ follows student's t-distribution with $K-1$ degrees of freedom. It suffices to show that

$$\left| P\left(\psi(\Sigma_{T,K}^{1/2}\xi) \leq a \right) - P\left(\psi(\Sigma_K^{1/2}\xi) \leq a \right) \right| = O(T^{-1}) \quad \forall a \in \mathbb{R}.$$

Clearly, $\psi(\cdot)$ is Lipschitz. Since the student's t-distribution with $K-1$ degrees of freedom has bounded density, we only need to show that $\|\Sigma_{T,K}^{1/2}\xi - \Sigma_K^{1/2}\xi\|_2 = O_P(T^{-1})$.

Now we do so by showing that $\Sigma_{T,K} - \Sigma_K = O(T^{-1})$. The diagonal entries in both matrices are the same. We consider the off-diagonal entries. Fix $k, l \in \{1, ..., K\}$ with $l > k \geq 1$. Then

$$\left|(\Sigma_{T,K})_{k,l}\right|$$

$$= G \left|E(\hat{\beta}_k \hat{\beta}_l)\right|$$

$$= G^{-1} \left|\sum_{t_1 \in H_k} \sum_{t_2 \in H_l} E u_{t_1} u_{t_2}\right|$$

$$= G^{-1} \left|\sum_{t_1=(k-1)G+1}^{kG} \sum_{t_2=(l-1)G+1}^{lG} E u_{t_1} u_{t_2}\right|$$

$$= G^{-1} \left|\sum_{t_1=(k-1)G+1}^{kG} \sum_{t_2=(l-1)G+1}^{lG} \gamma(t_2 - t_1)\right|$$

$$= G^{-1} \left|\sum_{t_1=(k-1)G+1}^{kG} \sum_{h=(l-1)G+1-t_1}^{lG-t_1} \gamma(h)\right|$$

$$= G^{-1} \left|\sum_{h=(l-2)G+1}^{(l-1)G} \gamma(h) \left[\min\left\{(l-k+1)G - h, G\right\} - \max\left\{(l-k)G + 1 - h, 1\right\}\right]\right|$$

$$\leq G^{-1} \sum_{h=(l-2)G+1}^{(l-1)G} |\gamma(h)| \cdot |[\min\left\{(l-k+1)G - h, G\right\} - \max\left\{(l-k)G + 1 - h, 1\right\}]|$$

$$\leq G^{-1} \sum_{h=(l-2)G+1}^{(l-1)G} |\gamma(h)|h \leq KT^{-1} \sum_{h=1}^{\infty} |\gamma(h)|h.$$

Since $\sum_{h=1}^{\infty} |\gamma(h)|h$ is bounded, we have $(\Sigma_{T,K})_{k,l} = O(T^{-1})$. Therefore, $\Sigma_{T,K} - \Sigma_K = O(T^{-1})$. The proof is complete. $\qquad\square$

G.13 Proof of Corollary 1

By Theorem 2, we have that

$$P\left(\mathbb{T}_K \in [-t_{K-1}(1-\alpha/2), t_{K-1}(1-\alpha/2)]\right) \to 1 - \alpha.$$

Note that the event

$$\{\mathbb{T}_K \in [-t_{K-1}(1-\alpha/2), t_{K-1}(1-\alpha/2)]\}$$

is equivalent to the event

$$\left\{\tau \in \left[\hat{\tau} - t_{K-1}(1-\alpha/2)\frac{\hat{\sigma}_{\hat{\tau}}}{\sqrt{K}}, \ \hat{\tau} + t_{K-1}(1-\alpha/2)\frac{\hat{\sigma}_{\hat{\tau}}}{\sqrt{K}}\right]\right\}.$$

Thus the result follows. □

H   Additional simulations

H.1   Performance with large T_0

In this section, we explore the performance of the t-test when $T_0 = 150$. These simulations are motivated by the theoretical results in Section 4.2, which require that T_0 is much larger than T_1. A larger T_0 further allows us to explore the impact of choosing a larger K.

Table 7 shows the results for $(T_0, T_1, N) = (150, 16, 14)$ and $K \in \{4, 6, 8\}$. The results for DGP6–DGP7 confirm the theoretical results in Section 4.2, and the results for DGP8–DGP9 suggest that the t-test remains quite robust in settings not covered by our theory, provided that T_0 is large enough. Table 7 also shows that increasing K to $K = 6$ and $K = 8$ reduces the length of the confidence intervals, especially under misspecification, while not affecting coverage accuracy much.

Table 7: Simulation results with $(T_0, T_1, N) = (150, 16, 14)$

|	Bias×10			Coverage			Avg. length CI		
	$K = 4$	$K = 6$	$K = 8$	$K = 4$	$K = 6$	$K = 8$	$K = 4$	$K = 6$	$K = 8$
DGP1	-0.00	-0.00	-0.00	0.90	0.90	0.89	0.06	0.05	0.05
DGP2	0.00	0.00	-0.00	0.89	0.91	0.90	0.06	0.05	0.05
DGP3	0.01	0.01	0.02	0.89	0.90	0.90	0.48	0.41	0.38
DGP4	-0.01	-0.00	-0.02	0.90	0.90	0.91	0.48	0.41	0.39
DGP5	0.00	0.01	-0.01	0.91	0.91	0.91	0.48	0.42	0.39
DGP6	-0.01	-0.01	-0.02	0.89	0.88	0.87	0.06	0.05	0.05
DGP7	0.00	0.00	0.00	0.89	0.89	0.89	0.06	0.05	0.05
DGP8	0.01	0.01	0.00	0.86	0.85	0.84	0.41	0.35	0.32
DGP9	-0.01	-0.00	0.00	0.83	0.83	0.82	0.09	0.08	0.08

Notes:. Simulations are based on 10,000 repetitions. CI: Confidence interval. Nominal coverage: $1 - \alpha = 0.9$. The DGPs are described in Section 5.

H.2   Performance with small T_1

Figures 5 and 6 display the coverage and average length of the confidence intervals based on the t-test for DGP1–DGP9 when $T_0 = 30$ and $T_1 \in \{10, 12, 14, 16\}$ (recall that $T_1 = 16$ in the main text). Given the small sample setting, we choose $K = 3$. Overall, the performance of the t-test does not change much if the sample size decreases from $T_1 = 16$ to $T_1 = 10$, except

under DGP8, which is not covered by our theory. This finding is not surprising given that the persistence in the prediction errors is relatively low ($\rho_u = 0.31$) and changing T_1 does not affect the estimation of the weights. Under DGP8, and to a lesser extent under DGP6, coverage decreases as T_1 increases. This result is consistent with our theoretical results, which require T_0 to be much larger than T_1 when there are deviations from a common nonstationarity.

Figure 5: Coverage with small T_1
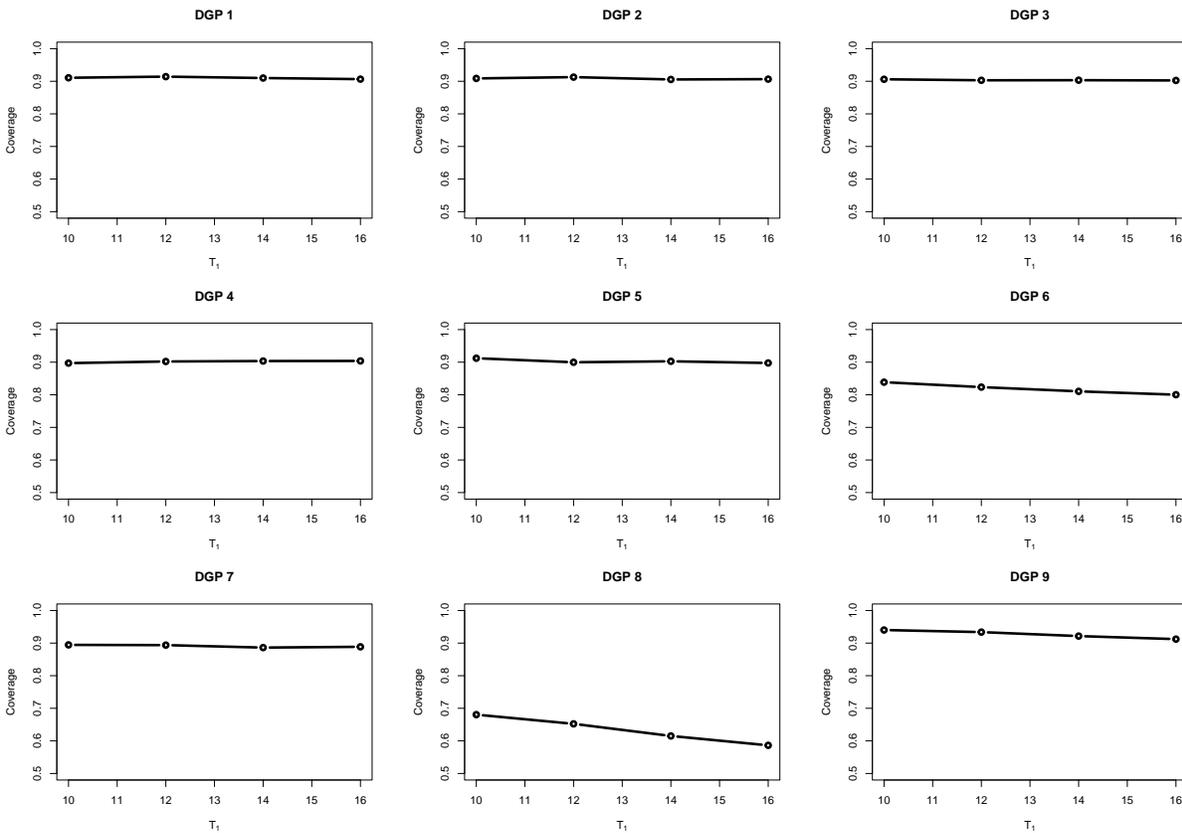
Notes: Simulations with 10,000 repetitions. Nominal coverage: $1 - \alpha = 0.9$. The DGPs are described in Section 5.

Figure 6: Length with small T_1
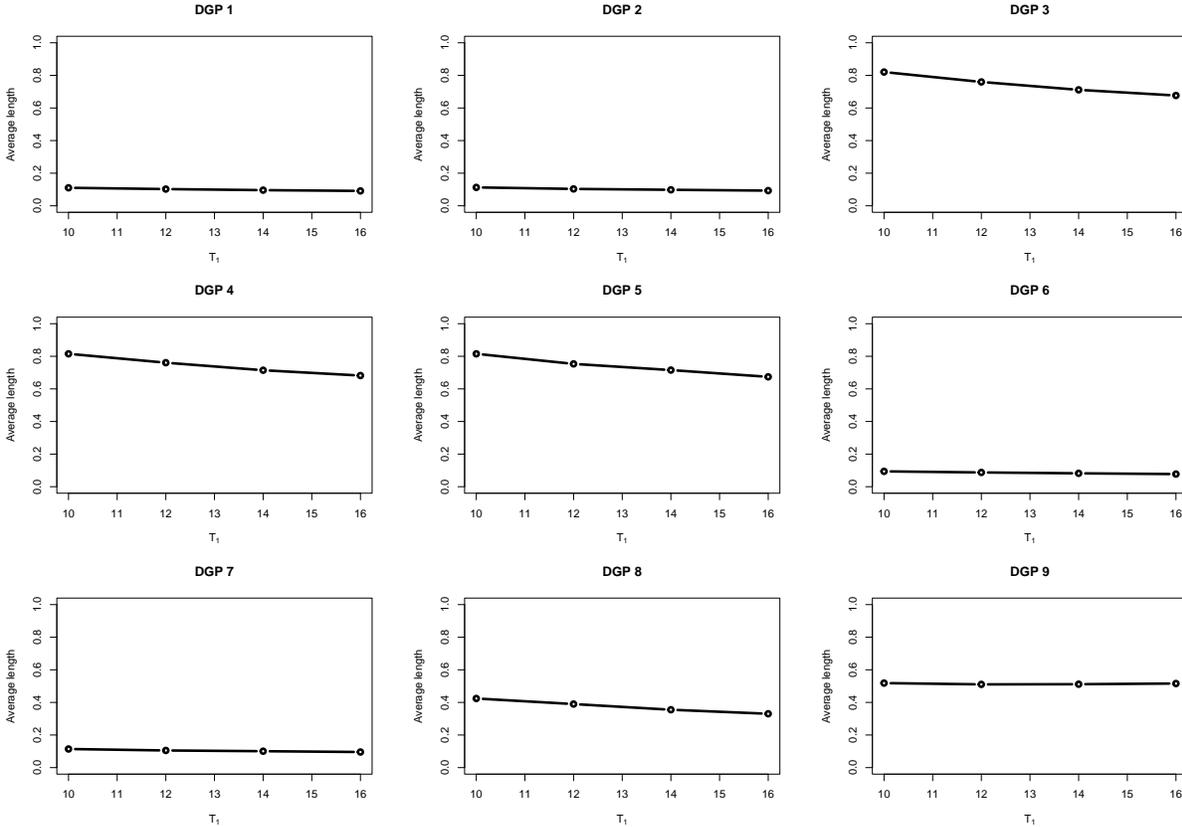
Notes: Simulations with 10,000 repetitions. Nominal coverage: $1 - \alpha = 0.9$. The DGPs are described in Section 5.

H.3   Comparison to permutation method of Abadie et al. (2010)

The permutation approach of Abadie et al. (2010) is a popular inference method in SC applications. It is fundamentally different from the t-test. First, it is designed to test sharp null hypotheses, such as the null of no effect whatsoever, whereas the t-test is designed to make inferences on the ATT. Second, it is typically motivated from a design-based perspective and requires random assignment of the treatment to be valid, whereas the t-test is developed within a sampling-based framework and allows for selection on observables and unobservables (see Appendix A).

Given these differences, we focus on size and power in a setting with constant effects to compare both methods. We implement the permutation test as described in Section 3.5 of Abadie (2021). Figure 7 displays the empirical rejection probabilities for both tests over a grid of alternatives $\alpha_t = a$ for $t \geq T_0$. We consider the same DGPs as in the main text and two different sample sizes $(T_0, T_1, N) = (30, 16, 14)$ ("small") and $(T_0, T_1, N) = (150, 16, 14)$

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("large"). In light of our simulation results, we choose $K = 4$ for the small sample case and $K = 8$ for the large sample case.

The simulation results show that the t-test is more powerful than the permutation approach overall, despite testing a weaker null hypothesis. The permutation test is only comparable to the t-test in terms of power under stationarity and correct specification (DGP1 and DGP2) *and* when $T_0 = 30$. For all the other DGPs and when $T_0 = 150$, the t-test is more powerful, and the permutation test can have essentially no power in some cases. We note that while the permutation test typically underrejects under our specific DGPs, Hahn and Shi (2017) show that this test can also substantially overreject under DGPs with sampling-based variation.

Figure 7: Comparison to permutation approach
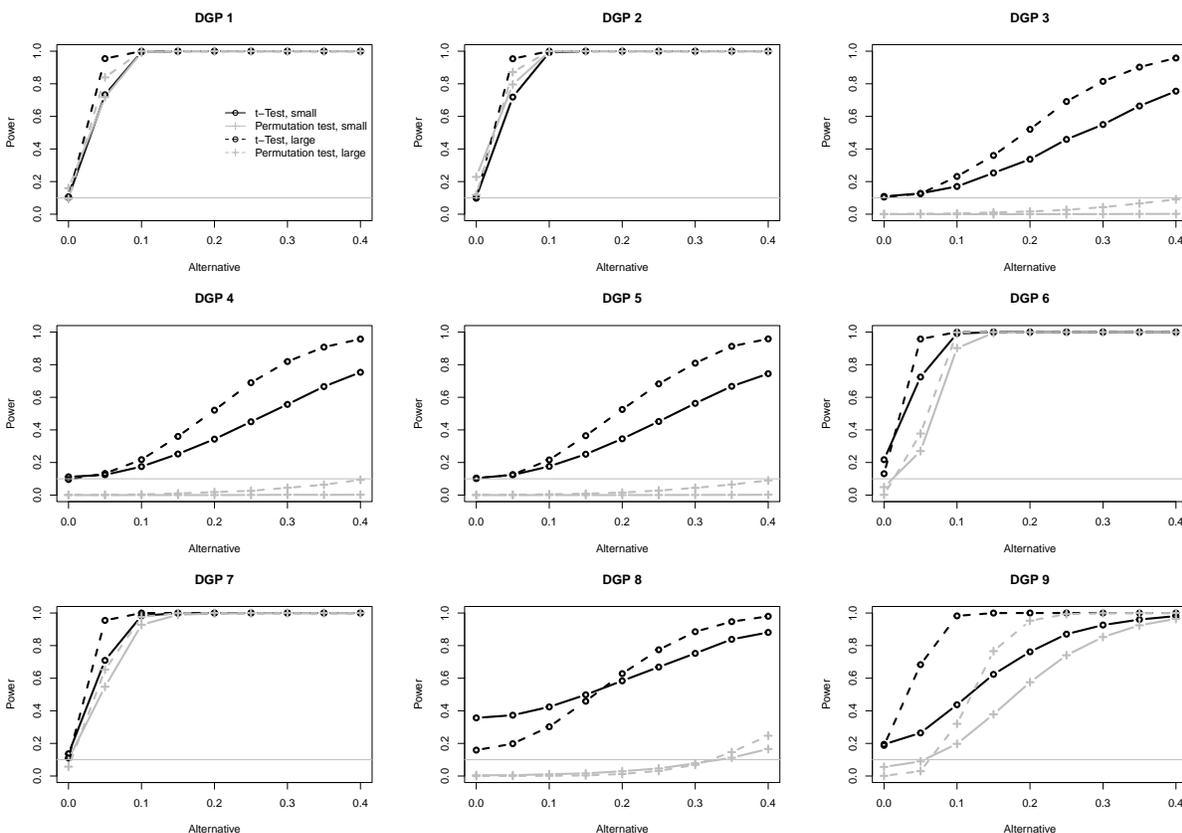
Notes: Simulations with 10,000 repetitions. Nominal level: $\alpha = 0.1$. Small: $(T_0, T_1, N) = (30, 16, 14)$ and $K = 4$ for the t-test. Large: $(T_0, T_1, N) = (150, 16, 14)$ and $K = 8$ for the t-test. The DGPs are described in Section 5.