Planar Packing of Diameter-Four Trees

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Abstract

We prove that, for every two n-node non-star trees of diameter at most four, there exists an n-node planar graph containing them as edge-disjoint subgraphs.

1 Introduction

The *packing problem* is to find an n-node graph G containing given n-node graphs $G_1, \dots, G_k$ as edge-disjoint subgraphs. Such a problem has been studied within a wide range of variants (see, e.g., [1, 3, 2]). A special attention has been devoted to packing of trees. Hedetniemi [6] showed that any two n-node non-star trees can be packed in K_n. A *star* is a tree with one node of degree $n-1$ and $n-1$ *leaves*. Maheo *et al.* [7] characterize which triples of trees can be packed in K_n.

Let $G_1, \dots, G_k$ be n-node planar graphs. A *planar packing* of $G_1, \dots, G_k$ is an n-node planar graph containing all the G_i's as edge-disjoint subgraphs. In [5] García *et al.* exhibited the following:

Conjecture 1 [5] *Any two trees different from a star admit a planar packing.*

The hypothesis that each tree is different from a star is necessary, since any mapping between the nodes of a star and the nodes of any tree leads to common edges. In the following, unless otherwise specified, we assume all considered trees to be different from a star.

García *et al.* proved the conjecture if the trees are *isomorphic* or if one of the trees is a *path*. In [8], Oda and Ota proved that the conjecture holds if one of the trees is a *caterpillar*, i.e., a tree which becomes a path when all its leaves are deleted, or if one of the trees is a diameter-four *spider tree*, where a spider tree has at most one node of degree greater than two and the diameter of a tree T is the maximum number of edges in a simple path in T. In [4], an algorithm is presented for constructing a planar packing of any tree and any spider tree. In this paper we show the following:

Theorem 1 *There exists a planar packing of every two non-star trees of diameter at most four.*

Small-diameter trees have a simple topology. However, as noticed in [8], they are an appealing case to study, as they are *star-like* trees, hence they are among the trees more likely to provide a counter-example to Conjecture 1, if any such a counter-example exists.

2 Proof of Theorem 1

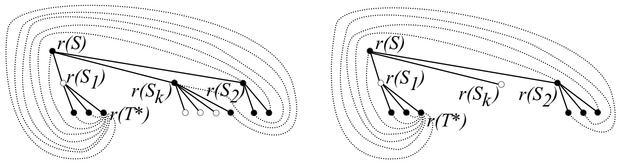
Let $|R|$ denote the number of nodes in a tree R and let $r(R)$ denote the *root* of a *rooted tree* R, where a rooted tree is a tree with one distinguished node, called root. Let T and S be two n-node trees of diameter at most four. Root them at nodes $r(T)$ and $r(S)$, respectively, so that their *height*, i.e. the maximum number of edges in any path from the root to a leaf, is at most two. Suppose, w.l.o.g., that the subtree T^* of T with the greatest number of nodes has at least as many nodes as the subtree of S with the greatest number of nodes.

In the following, whenever we say that we embed S in the plane, we assume to embed it *downward*, i.e., with every node below its parent. This allows us to speak of a left-to-right order of the children of each non-leaf node of S. Once fixed the embedding of S, we refer to a total left-to-right order of the nodes of S so that $r(S)$ is the first node, the root of the leftmost subtree of $r(S)$ is the second node, its children come in left-to-right order after their parent, the root of the second leftmost subtree of $r(S)$ is the next node, its children come in left-to-right order after their parent, and so on.

Overall Strategy. A planar packing of T and S is constructed by progressively embedding S in the plane and by embedding T *over* S, i.e., by mapping the nodes of T to embedded nodes of S and by then routing the edges of T. At any step of the construction, denote by $\mathcal{E}$ the partially constructed planar packing of T and S. At any step of the construction, a node of S is *free* if it is embedded in the plane and no node of T has yet been mapped to it. The algorithm distinguishes two cases. In Case 1, it is possible to embed T^* over some embedded subtrees of S, in such a way that a node of T^* is mapped to $r(S)$ and there is a free node in one of the embedded subtrees of S. Node $r(T)$ is possibly mapped to such a free node. In Case 2, it is not possible to compute such an embedding of T^* over some subtrees of $r(S)$. However, such a condition implies very strict constraints on the topology of S.

Now we describe in detail the two main cases of the algorithm. In *Case 1*, there exists a subtree S^0 of

S such that $|S^0| = 1$, or (ii) there exists a sequence $\mathcal{S} = (S_1, \cdots, S_k)$ of subtrees of $r(S)$ such that $|S_1| > 1$, $\sum_{i=1}^{k} |S_i| > |T^*|$, and $\sum_{i=1}^{k-1} |S_i| < |T^*|$. If neither (i) nor (ii) holds, then we are in *Case 2*.



(a) (b)

Figure 1: Embedding T^* over a sequence $\mathcal{S} = (S_1, \cdots, S_k)$ of subtrees of $r(S)$ such that: (a) $|S_1| > 1$, $\sum_{i=1}^{k-1} |S_i| < |T^*|$, and $\sum_{i=1}^{k} |S_i| > |T^*|$; (b) $|S_1| > 1$, $\sum_{i=1}^{k-1} |S_i| = |T^*|$, and $|S_k| = 1$. Black circles correspond to nodes of T^* mapped to nodes of S. White circles are free nodes of S. In these examples $k = 3$.

Case 1. See Fig. 1. Suppose that (i) holds. Then, there exists a subtree S^0 of $r(S)$ such that $|S^0| = 1$. Let $\mathcal{S}$ be a sequence of subtrees of $r(S)$ to be determined. Embed any subtree $|S_1|$ of S such that $|S_1| > 1$ as the leftmost subtree of $r(S)$. Such a subtree exists, as S is different from a star. Place $r(T^*)$ at the rightmost child of $r(S_1)$. Place children of $r(T^*)$ at the other children of $r(S_1)$, if any, and place a child of $r(T^*)$ at $r(S)$. The number of children of $r(T^*)$ is sufficient to *cover* all such nodes of S, since T^* has at least as many nodes as any subtree of $r(S)$. We embed some subtrees of $r(S)$ different from S^0 as the rightmost subtrees of $r(S)$, starting from the rightmost subtree S_2, then embedding S_3 to the left of S_2, then embedding S_4 to the left of S_3, and so on. We choose such subtrees in any way. We stop at a subtree S_x such that $\sum_{j=1}^{x} |S_j| \geq |T^*|$ and $\sum_{j=1}^{x-1} |S_j| < |T^*|$. For $j = 2, \cdots, x$, when S_j is embedded to the left of S_{j-1} (or as the rightmost subtree of $r(S)$ if $j = 2$), children of $r(T^*)$ are mapped first to $r(S_j)$, and then to all the children of $r(S_j)$, in right-to-left order. Since $\sum_{j=1}^{x-1} |S_j| < |T^*|$, each node in S_j, with $j = 2, \cdots, x-1$, has a node of T^* mapped to it; if $x \geq 2$, a node of T^* is mapped to $r(S_x)$ and, possibly, to some children of $r(S_x)$. If there is a child of $r(S_x)$ with no node of T^* mapped to it ($\sum_{j=1}^{x} |S_j| > |T^*|$), then define $\mathcal{S} = (S_1, \cdots, S_x)$. Otherwise ($\sum_{j=1}^{x} |S_j| = |T^*|$), nodes of T^* have been mapped to all nodes of S_x; embed S^0 to the left of S_x and define $\mathcal{S} = (S_1, \cdots, S_x, S^0)$. Route the edges of T^* in clockwise direction around the partially constructed embedding of S.

If (ii) holds, then we already assume to have a sequence $\mathcal{S} = (S_1, \cdots, S_k)$ of subtrees of $r(S)$ such that $|S_1| > 1$, $\sum_{i=1}^{k} |S_i| > |T^*|$, and $\sum_{i=1}^{k-1} |S_i| < |T^*|$. An embedding of T^* and of the subtrees in $\mathcal{S}$ can be constructed as above.

Let $\mathcal{S} = (S_1, \cdots, S_k)$ be the sequence of embedded subtrees of $r(S)$. Complete the embedding of S by inserting the subtrees of $r(S)$ not in $\mathcal{S}$ in any order to the right of S_1 and to the left of S_k. Denote by $S_1^*, \cdots, S_l^*$ such subtrees in left-to-right order.

Order the subtrees of T different from T^* in any way $T_1, \cdots, T_m$. Embed T_1 with $r(T_1)$ on $r(S_1)$, and with the x children of $r(T_1)$ mapped to the first x free nodes of S in left-to-right order (see Fig. 2). The edges connecting $r(T_1)$ with its children are routed in counter-clockwise direction around S.

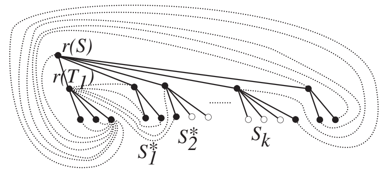


Figure 2: Embedding T_1 over S.

Now embed subtrees $T_2, \cdots, T_m$ one at a time, while maintaining the following invariants. Suppose that $T_2, \cdots, T_{j-1}$ have been embedded and $T_j, \cdots, T_m$ still have to be embedded.

Invariant A: There exists a free node n_r of S_k such that leaves of T have been mapped to all the neighbors of n_r and to all nodes to the right of n_r.

Invariant B: Nodes $r(T^*), r(T_1), \cdots, r(T_{j-1})$ and all free nodes of S are on the outer face of $\mathcal{E}$. Further, none of $r(T^*), r(T_1), \cdots, r(T_{j-1})$ is to the right of a free node of S.

Invariant C: If the leftmost free node n_f of S is a node $r(S_i^*)$ then all nodes in $S_i^*, \cdots, S_l^*$ are free.

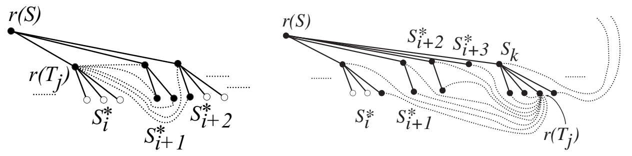
The invariants hold after T^* and T_1 have been embedded. In particular, the choice of $\mathcal{S} = (S_1, \cdots, S_k)$ and the embedding of T^* were done in such a way that Invariant A is satisfied.

Before embedding T_j, three cases are possible.

Case 1.1: The leftmost free node n_f of S is a node of S_k. By Invariant A, nodes of T have been mapped to all nodes in $S_{k-1}, \cdots, S_2$. Then, the only free nodes are children of $r(S_k)$ and, by Invariant B, they are on the outer face of $\mathcal{E}$. Embed each of $T_j, \cdots, T_m$ into consecutive free children of $r(S_k)$, with the root of each subtree to the left of its children. Place $r(T)$ at the last free child n_r of $r(S_k)$ and draw edges connecting it to $r(T^*), r(T_1), \cdots, r(T_m)$. Such edges can be routed without crossings since, by Invariant B, all such nodes are on the outer face of $\mathcal{E}$. Moreover, no common edge is inserted since, by Invariant A, a leaf of T has been mapped to the only neighbor of n_r.

Case 1.2: The leftmost free node n_f of S is a node $r(S_i^*)$. By Invariant C, all nodes in $S_i^*, \cdots, S_l^*$ are free. Let f be the number of free nodes in S_k.

If $|T_j| \leq |S_{i+1}^*| + \cdots + |S_l^*| + f$ (see Fig. 3.a), embed T_j with $r(T_j)$ on $r(S_i^*)$, and with the x children of



(a) (b)

Figure 3: Embedding T_j when the leftmost free node of S is a node $r(S_i^*)$ and when: (a) $|T_j| \leq |S_{i+1}^*| + \cdots + |S_l^*| + f$; (b) $|T_j| > |S_{i+1}^*| + \cdots + |S_l^*| + f$.

$r(T_j)$ mapped to the first x free nodes in left-to-right order starting from $r(S_{i+1}^*)$. Counter-clockwise route the edges from $r(T_j)$ to its children around S from the last occurrence of $r(T_j)$ on the border of $\mathcal{E}$. It is easy to see that no crossing is introduced, that no common edge is inserted, and that Invariants A–C are maintained. In particular, since $|T_j| \leq |S_{i+1}^*| + \cdots + |S_l^*| + f$, there is still a free node in S_k as required by Invariant A.

If $|T_j| > |S_{i+1}^*| + \cdots + |S_l^*| + f$ (see Fig. 3.b), map $r(T_j)$ to the righmost free node in S_k, map its children to all free nodes in S_k, to all free nodes in $S_{i+1}^*, \cdots, S_l^*$, to $r(S_i^*)$, and, if there are still children of $r(T_j)$ to embed, to the righmost children of $r(S_i^*)$. The only free nodes remaining are children of $r(S_i^*)$. Hence, $T_{j+1}, \cdots, T_m$ can be embedded into consecutive free children of $r(S_i^*)$, with the root of each subtree to the left of its children. Place $r(T)$ at the last free child of $r(S_i^*)$ and draw edges connecting it to $r(T^*), r(T_1), \cdots, r(T_m)$. Such edges can be routed without crossings since, by Invariant B, all such nodes are on the outer face of $\mathcal{E}$. Further, a leaf of T has been mapped to $r(S_i^*)$, hence no common edge is inserted.

Case 1.3: The leftmost free node n_f of S is a child of a node $r(S_i^*)$. Place $r(T_j)$ at n_f. Map the x children of $r(T_j)$ to the next x free nodes to the right of n_f. Route the edges from n_f in counter-clockwise direction around S. It is easy to see that no crossing is introduced, that no common edge is inserted, and that Invariants A–C are maintained. In particular, the edges from $r(T_j)$ to its children leave on the outer face all nodes $r(T^*), r(T_1), \cdots, r(T_{j-1})$, which, by Invariant B, are to the left of n_f.

When the last subtree of $r(T)$ has been embedded, if the above described algorithm has not already mapped $r(T)$ to a node of S, then map $r(T)$ to the only remaining free node n_r of S. By Invariant A, n_r belongs to S_k. Route edges from $r(T)$ to all its children. Such edges do not cause crossings, since, by Invariant B, all roots of subtrees of $r(T)$ are on the outer face of $\mathcal{E}$. Further, by Invariant A, a leaf of T has been mapped to the only neighbor of n_r, hence no common edge is inserted.

Case 2. In *Case 2* no subtree of $r(S)$ has only one node and there exists no sequence $\mathcal{S} = (S_1, \cdots, S_k)$ of subtrees of $r(S)$ such that $\sum_{i=1}^{k} |S_i| > |T^*|$ and such that $\sum_{i=1}^{k-1} |S_i| < |T^*|$. We observe the following:

Lemma 2 *Let $\mathcal{U} = \{u_1, \cdots, u_k\}$ be a multiset of positive integers and let x be a positive integer. Suppose that $\sum_{i=1}^{k} u_i > x$. Then, there exists an ordering $\mathcal{O}$ of $\mathcal{U}$ such that no initial subsequence of $\mathcal{O}$ sums up to x if and only if there exists no integer c such that $u_i = x/c$ holds for all $i = 1, \cdots, k$.*

Proof. Suppose that all integers u_i are such that $u_i = x/c$, for some integer c. Consider any ordering $\mathcal{O}$ of $\mathcal{U}$. The first c elements of $\mathcal{O}$ sum up to $c(x/c) = x$.

Suppose that there exists no integer c such that $u_i = x/c$ holds for all $i = 1, \cdots, k$. Then, either all integers u_i are equal to an integer y, that is not a divisor of x, or there exists at least two distinct integers in $\mathcal{U}$. In the first case, the first m elements of any ordering $\mathcal{O}$ of $\mathcal{U}$ sum up to $my \neq x$, for each $1 \leq m \leq k$. In the second case, we show how to construct an ordering $\mathcal{O}$ of $\mathcal{U}$ such that no initial subsequence of $\mathcal{O}$ sums up to x. We insert integers into $\mathcal{O}$ in several steps. At the beginning of the j-th step, the following invariants hold: (1) the set $\mathcal{U}_j \subseteq \mathcal{U}$ of integers that have not yet been inserted into $\mathcal{O}$ contains at least two distinct integers; (2) the sum s_j of the elements that have been inserted in $\mathcal{O}$ is less than x. Notice that such invariants hold at the beginning of the first step, i.e., when no element has yet been inserted in $\mathcal{O}$. The j-th step inserts elements at the end of $\mathcal{O}$ as in the following cases:

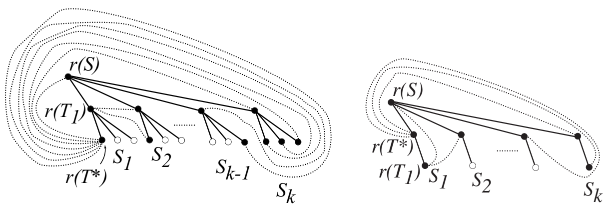
Case A: If an integer u^* exists in U_j such that $s_j + u^* > x$, then let it be the next element of $\mathcal{O}$. Insert the integers in $\mathcal{U}_j \setminus \{u^*\}$ in whichever order after u^*, completing a sequence $\mathcal{O}$ which satisfies the requirements of the lemma.

Case B: If Case A does not apply, and if there exists an integer u^* in U_j such that $s_j + u^* < x$ and such that $\mathcal{U}_j \setminus \{u^*\}$ contains at least two distinct integers, then let u^* be the next element of $\mathcal{O}$. At the $(j+1)$-th step, $\mathcal{U}_{j+1} = \mathcal{U}_j \setminus \{u^*\}$ and $s_{j+1} = s_j + u^*$ satisfy invariants (1) and (2) by hypothesis.

Case C: If Cases A and B do not apply, consider two distinct integers u_1^* and u_2^* in $\mathcal{U}_j$. Both of them are less than or equal to $x - s_j$, otherwise Case A would apply; since u_1^* and u_2^* are distinct, one of them, say u_1^*, is *less than* $x - s_j$. Hence, $\mathcal{U}_j$ contains no element with the same value of u_1^* and no element with value different from u_1^* and u_2^*, otherwise Case B would apply. Thus, all elements of $\mathcal{U}_j \setminus \{u_1^*, u_2^*\}$, if any, are equal to u_2^*. If there is no element in $\mathcal{U}_j \setminus \{u_1^*, u_2^*\}$, then, by the hypotheses of the lemma, $s_j + u_1^* + u_2^* > x$, hence inserting u_1^* and u_2^* in this order in $\mathcal{O}$ yields $\mathcal{O}$ to satisfy the requirements of the lemma. Otherwise, there are some elements with the same value of u_2^* in $\mathcal{U}_j \setminus \{u_1^*, u_2^*\}$. It follows that u_2^* cannot be less than $x - s_j$, otherwise Case B would apply. Hence $u_2^* = x - s_j$. Inserting u_1^* as next element in $\mathcal{O}$ and u_2^* after u_1^* again leads to a sequence $\mathcal{O}$ that satisfies the requirements of the lemma. □

Suppose we are in *Case 2*. Consider the multiset $\mathcal{U}$ with an integer u_i such that $|S_i| = u_i$ for each subtree S_i of S. Since no subtree S^0 of $r(S)$ exists such that $|S^0| = 1$, then $u_i > 1$, for all $u_i \in \mathcal{U}$. By Lemma 2 with $x = |T^*|$ (notice that $\sum_i u_i = \sum_i |S_i| = n-1 > n-2 \geq |T^*|$), an ordering $\mathcal{O}$ of $\mathcal{U}$ exists such that no initial subsequence of $\mathcal{O}$ sums up to x if and only if no integer c exists such that $u_i = x/c$ holds for all $i = 1, \cdots, k$. Hence, if we are in *Case 2*, there exists an integer c such that $|S_i| = |T^*|/c$ holds for all subtrees $S_1, \cdots, S_k$ of $r(S)$. Downward embed the subtrees of $r(S)$ in any left-to-right order $S_1, \cdots, S_k$. Notice that $k > 1$ and that $|S_i| > 1$, otherwise S would be a star. With respect to *Case 1*, we embed T^* in a different way. Namely, we further distinguish two cases.

(a) Refer to Fig. 4.a. If $|S_i| > 2$, map $r(T^*)$ to the leftmost child of $r(S_1)$. Map its children to $r(S)$, to $r(S_k)$, to the children of $r(S_k)$ in right-to-left order, to $r(S_{k-1})$, to the children of $r(S_{k-1})$ in right-to-left order, and so on while there are children of $r(T^*)$ to embed. $(c-1)|S_i| = |T^*| - |S_i|$ nodes of T^* cover subtrees $S_k, S_{k-1}, \cdots, S_{k-c+2}$ (if $c = 1$, then no subtree of $r(S)$ is entirely covered by T^*). Two nodes of T^* are mapped to $r(S)$ and the leftmost child of $r(S_1)$. Hence, $|S_i| - 2$ children of $r(T^*)$ are placed on subtree S_{k-c+1}. Since $|S_i| \geq 3$, then $r(S_{k-c+1})$ is covered by a node of T^* and the leftmost $|S_i| - (|S_i| - 2) = 2$ children of $r(S_{k-c+1})$ are free. Embed a subtree $T_1 \neq T^*$ of $r(T)$ as in Case 1.2 (notice that the leftmost free node of S is $r(S_1)$). Now, Invariants A–C of the algorithm described for Case 1 hold (notice that the subtrees of $r(S)$ have different names with respect to those in Invariants A–C) and an embedding of T on S can be completed by repeatedly applying Cases 1.1, 1.2, and 1.3.



(a) (b)

Figure 4: Illustrations for *Case 2(a)* and *Case 2(b)*.

(b) Refer to Fig. 4.b. If $|S_i| = 2$, map $r(T^*)$ to $r(S_1)$. Map the children of $r(T^*)$ to $r(S_k)$, to the child of $r(S_k)$, to $r(S_{k-1})$, to the child of $r(S_{k-1})$, and so on while there are children of $r(T^*)$ to embed. $S_k, S_{k-1}, \cdots, S_{k+2-(|T^*|/2)}$ are covered by nodes of T^* (if $|T^*| = 2$, then no subtree of $r(S)$ is entirely covered by T^*) and a node of T^* is also mapped to $r(S_{k+1-(|T^*|/2)})$, while the child of $r(S_{k+1-(|T^*|/2)})$ is free. If T^* is the only subtree of $r(T)$ with more than one node, then map $r(T)$ to the child of $r(S_{k+1-(|T^*|/2)})$ and all its children different from $r(T^*)$ to the other free nodes of S. Draw edges from $r(T)$ to its children. Since all such children are on the outer face of $\mathcal{E}$, the drawn edges do not cause crossings. If there exist at least two subtrees T^* and T_1 of $r(T)$ with more than one node, then map $r(T_1)$ to the child of $r(S_1)$ and map the children of $r(T_1)$ to $r(S)$ and to the leftmost $|T_1| - 2$ free nodes of S. Now, Invariants A–C of the algorithm described for Case 1 hold (notice that the subtrees of $r(S)$ have different names with respect to those in Invariants A–C) and an embedding of T on S can be completed by repeatedly applying Cases 1.1, 1.2, and 1.3.

3 Conclusions

In this paper we described how to obtain a planar packing of any two non-star trees of diameter at most four. The algorithm we presented can be implemented to run efficiently. Our algorithm uses a *divide et impera* strategy, namely distinct subtrees of a tree T are mapped to distinct forests of a tree S, intermixed with some counting arguments. We believe worth of further research efforts understanding whether the design of similar (and more involved) recursive algorithms, that maintain some strong topological invariants while being guided by some counting arguments, can lead to prove the planar-packing conjecture.

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