AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **34** (2006), Pages 69–87

On optimal orientations of tree vertex-multiplications

K. M. Koh

Department of Mathematics
National University of Singapore
2 Science Drive 2
Singapore 117543

E. G. Tay

Mathematics and Mathematics Education
National Institute of Education
Nanyang Technological University
1 Nanyang Walk
Singapore 637616

Abstract

For a bridgeless connected graph G, let $\mathcal{D}(G)$ be the family of strong orientations of G; and for any $D \in \mathcal{D}(G)$, we denote by $d(D)$ (resp., $d(G)$) the diameter of D (resp., G). Define $\vec{d}(G) = \min\{d(D) | D \in \mathcal{D}(G)\}$. In this paper, we study the problem of evaluating $\vec{d}(T(s_1, s_2, \ldots, s_n))$, where $T(s_1, s_2, \ldots, s_n)$ is a T vertex-multiplication for any tree T of order $n \geq 4$ and diameter at least 3, and any sequence (s_i) with $s_i \geq 2$, $i = 1, 2, \ldots, n$. We show that $\vec{d}(T(s_1, s_2, \ldots, s_n)) \leq d(T)+1$ with $\vec{d}(T(s_1, s_2, \ldots, s_n)) = d(T)$ for most cases.

1 Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the **eccentricity** $e(v)$ of v is defined as $e(v) = \max\{d(v, x) | x \in V(G)\}$, where $d(v, x)$ denotes the distance from v to x. The **diameter** of G, denoted by $d(G)$, is defined as $d(G) = \max\{e(v) | v \in V(G)\}$. Let D be a digraph with vertex set $V(D)$ and edge set $E(D)$. For $v \in V(D)$, the notions $e(v)$ and $d(D)$ are similarly defined.

An *orientation* of a graph G is a digraph obtained from G by assigning to each edge in G a direction. An orientation D of G is *strong* if every two vertices in D are mutually reachable in D. An edge e in a connected graph G is a *bridge* if $G - e$ is disconnected. Robbins' celebrated one-way street theorem [25] states that a *connected graph G has a strong orientation if and only if no edge of G is a bridge.*

Efficient algorithms for finding a strong orientation for a bridgeless connected graph can be found in Roberts [26], Boesch and Tindell [1] and Chung et al. [2]. Boesch and Tindell [1] extended Robbin's result to mixed graphs where edges could be directed or undirected. Chung et al. [2] provided a linear-time algorithm for testing whether a mixed graph has a strong orientation and finding one if it does. As another possible way of extending Robbins' theorem, consider further the notion $\rho(G)$ given below (see Boesch and Tindell [1], Chvátal and Thomassen [3], and Roberts [27]). Given a connected graph G containing no bridges, let $\mathcal{D}(G)$ be the family of strong orientations of G. Define

$$\rho(G) = \min\{d(D) | D \in \mathcal{D}(G)\} - d(G).$$

The first term on the right side of the above equality is essential. Let us write

$$\vec{d}(G) = \min\{d(D) | D \in \mathcal{D}(G)\}.$$

The problem of evaluating $\vec{d}(G)$ for an arbitrary connected graph G is very difficult. As a matter of fact, Chvátal and Thomassen [3] showed that the problem of deciding whether a graph admits an orientation of diameter two is NP-hard.

On the other hand, the parameter $\vec{d}(G)$ has been studied in various classes of graphs including the cartesian product of graphs (Plesník [23], Soltés [32], McCanna [21], Roberts and Xu [28–31], Koh and Tan [8], Koh and Tay [11–17], Konig et al. [19]), complete graphs (Plesník [22], Boesch and Tindell [1], Maurer [20] and Reid [24]), complete bipartite graphs (Plesník [23], Boesch and Tindell [1], Soltés [32] and Gutin [5]) and complete n-partite graphs for $n \geq 3$ (Plesník [23], Gutin [5–7] and Koh and Tan [9, 10]). These optimal orientations can be used to provide optimal arrangements of one-way streets (Robbins [25], Roberts and Xu [28–31], and Koh and Tay [12]). They can also be used to solve a variant of the gossip problem on a graph G where all points simultaneously broadcast items to all other points in such a way that items are combined at no cost and all links are simultaneously used but in only one direction at a time. In this problem, the time taken for the gossip to be completed is bounded below by $d(G)$ and above by $\min\{2d(G), \vec{d}(G)\}$ (see Fraigniaud and Lazard [4]). Thus the problem for a graph G is solved completely if $\rho(G) = 0$.

In [18], Koh and Tay extended the results on the complete n-partite graphs by introducing a new family of graphs based on a given connected graph as follows. Let G be a given connected graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For any sequence of n positive integers (s_i), let $G(s_1, s_2, \ldots, s_n)$ denote the graph with vertex set V^* and edge set E^* such that $V^* = \bigcup_{i=1}^{n} V_i$, where V_i's are pairwise disjoint sets with $|V_i| = s_i$, $i = 1, 2, \ldots, n$; and for any two distinct vertices x, y in V^*, $xy \in E^*$ if and only if $x \in V_i$ and $y \in V_j$ for some $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ such that $v_i v_j \in E(G)$. Call the graph $G(s_1, s_2, \ldots, s_n)$ a ***G* vertex-multiplication**. Thus when $G = K_n$, the complete graph of order n, the graph $G(s_1, s_2, \ldots, s_n)$ is a complete n-partite graph. We call G a **parent graph** of a graph H if $H \cong G(s_1, s_2, \ldots, s_n)$ for some sequence (s_i) of positive integers.

For convenience, we sometimes write, for $i = 1, 2, \ldots, n$, $V_i = \{(p, i) | 1 \leq p \leq s_i\}$ and call (p, i) the pth vertex in V_i. Thus two vertices (p, i) and (q, j) in V^* are adjacent in $G(s_1, s_2, \ldots, s_n)$ if and only if $i \neq j$ and $v_i v_j \in E(G)$. For $s = 1, 2, \ldots$, we shall denote $G(s, s, \ldots, s)$ simply by $G^{(s)}$. Thus $G^{(1)} = G$, and it is understood that the number of s's in $G(s, s, \ldots, s)$ is equal to the order of G.

In this paper, we shall study the case when G is a tree. Since trees of diameter not exceeding 2 are parent graphs to complete bipartite graphs which have been completely solved, we shall only consider trees of diameter exceeding 2. It was shown in [18] that if $s_i \geq 2$ for each $i = 1, 2, \ldots, n$ where $n \geq 3$, then $d(G) \leq \vec{d}(G(s_1, s_2, \ldots, s_n)) \leq d(G) + 2$. From this fundamental result, all graphs of the form $G(s_1, s_2, \ldots, s_n)$, where $s_i \geq 2$ for $1 \leq i \leq n$, can now be classified into 3 classes $\mathcal{C}_i$ in the following natural way:

$$\mathcal{C}_i = \{G(s_1, s_2, \ldots, s_n) | \vec{d}(G(s_1, s_2, \ldots, s_n)) = d(G) + 1\}, i = 0, 1, 2.$$

From now on, we shall assume that $s_i \geq 2$ for $1 \leq i \leq n$. In the subsequent sections, we shall show that if T is a tree of order n and diameter exceeding 2, then $T(s_1, s_2, \ldots, s_n) \in \mathcal{C}_0 \cup \mathcal{C}_1$ with $T(s_1, s_2, \ldots, s_n) \in \mathcal{C}_0$ for most but not all cases.

2 Terminology and Notation

Let D be a digraph. A dipath (resp., dicycle) in D is simply called a path (resp., cycle) in D. For $X \subseteq V(D)$, the subdigraph of D induced by X is denoted by $D[X]$, or simply $[X]$, if there is no danger of confusion. Given $F \in \mathcal{D}(G(s_1, s_2, \ldots, s_n))$, let $\bigcup_{j \in J} F_{v_j} = F[\{(p, v_j) | 1 \leq p \leq s_j, \, j \in J\}]$.

Let A be a subdigraph of F. The eccentricity, outdegree and indegree of a vertex (p, i) in A are denoted respectively by $e_A((p, i))$, $s_A((p, i))$ and $\bar{s_A}((p, i))$. The subscript A is omitted if $A = F$.

A digraph D_1 is said to be isomorphic to a digraph D_2, written $D_1 \cong D_2$, if there is a bijection $\varphi : V(D_1) \to V(D_2)$ such that $uv \in E(D_1)$ if and only if $\varphi(u)\varphi(v) \in E(D_2)$.

For $x, y \in V(D)$, we write '$x \to y$' or '$y \leftarrow x$' if x is adjacent to y in D. Also, for $A, B \subseteq V(D)$, we write '$A \to B$' or '$B \leftarrow A$' if $x \to y$ in D for all $x \in A$ and for all $y \in B$. When $A = \{x\}$, we shall write '$x \to B$' or '$B \leftarrow x$' for $A \to B$.

For convenience, we shall denote a tree with diameter d by T_d.

For clarity, we introduce an alternative way of labeling the vertices of a tree. Let T_d have a planar representation as follows: Choose a path P in T_d of length d and draw it vertically. We call P the **main path**. Label the vertices on P from (1) to $(d + 1)$ starting with (1) at the top and the others numbered consecutively downwards. If there is no ambiguity, vertex (i) may simply be written as i. A branch from a vertex (v) on P whose label does not exceed $(\lceil \frac{d}{2} \rceil)$ is drawn to the right and upwards in such a way that the neighbours of (v) are placed from left to right at the same height as the vertex $(v - 1)$. A branch from a vertex (v) in P whose label exceeds $(\lceil \frac{d}{2} \rceil)$ is drawn to the right and downwards in such a way that the neighbours of (v) are placed from left to right at the same height as the vertex $(v + 1)$.

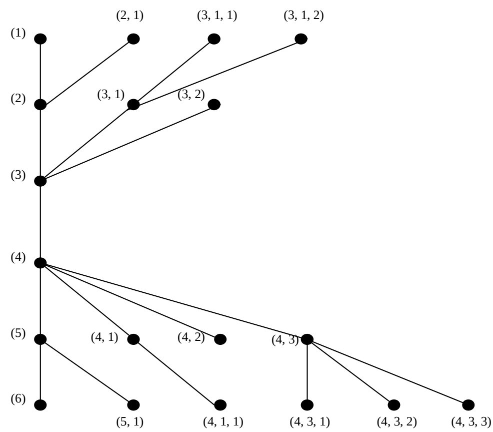

Figure 1

We shall now give an algorithm for labeling the vertices of T_d.

(i) The vertices of P from top to bottom have been labeled $(1), (2), \ldots, (d+1)$.

(ii) For $2 \leq i \leq d$, if $\deg((i)) = k_i \geq 3$, then label from left to right the unlabeled vertices adjacent to (i) as $(i,1), (i,2), \ldots, (i, k_i - 2)$.

(iii) Suppose a vertex v has been labeled $(a_1, a_2, \ldots, a_q)$. Label from left to right the unlabeled vertices adjacent to v as $(a_1, a_2, \ldots, a_q, 1)$, $(a_1, a_2, \ldots, a_q, 2)$, …, $(a_1, a_2, \ldots, a_q, \deg(v)\ -1)$.

For a vertex $v = (a_1, a_2, \ldots, a_m)$, define its **vertex number**, $n(v)$, as follows:

$$n(v) = \begin{cases} a_1 + 1 - m & \text{if} \quad a_1 \leq \lceil \frac{d}{2} \rceil; \\ a_1 + m - 1 & \text{otherwise.} \end{cases}$$

Denote by v^i the i-th coordinate in v.

As an illustration, the labeling of a tree T_d with $d = 5$ is shown in Figure 1.

For $i = 1, 2, \ldots, d$, we shall label the vertex v_i as (i) according to the labeling above.

3 Optimal orientations of $T_d(s_1, s_2, \ldots, s_n)$, where $d = 3, 4$

In this section, we shall obtain results on $T_d(s_1, s_2, \ldots, s_n)$, where $d = 3$ or 4. We shall need the following lemma to prove our results in this and the next section. The lemma has been proved in [18] but for completeness, we shall include the proof here.

Lemma 1 *Let t_i, s_i be integers such that $t_i \leq s_i$ for $1 \leq i \leq n$. If the graph $G(t_1, t_2, \ldots, t_n)$ admits an orientation F in which every vertex v lies on a cycle of length not exceeding m, then $\vec{d}\,(G(s_1, s_2, \ldots, s_n)) \leq \max\{m, d(F)\}$.*

Proof. Given such an orientation F of $G(t_1, t_2, \ldots, t_n)$, define an orientation F' of $G(s_1, s_2, \ldots, s_n)$ as follows:

(i) for $p < t_i$ and $q < t_j$, $(p, i) \to (q, j)$ iff $(p, i) \to (q, j)$ in F;

(ii) for $p < t_i$ and $q < t_j$, $(p, i) \to (q, j)$ iff $(p, i) \to (t_j, j)$ in F;

(iii) for $p < t_i$ and $q < t_j$, $(p, i) \to (q, j)$ iff $(t_i, i) \to (q, j)$ in F;

(iv) for $p < t_i$ and $q < t_j$, $(p, i) \to (q, j)$ iff $(t_i, i) \to (t_j, j)$ in F;

We shall now prove that $d(F') \leq \max\{m, d(F)\}$ by showing that for any 2 vertices (p, i) and (q, j) in F', $d((p, i), (q, j)) \leq \max\{m, d(F)\}$. Indeed, if $i \neq j$ or '$i = j$ and $p < t_i$ or $q < t_i$', then it is clear that $d((p, i), (q, j)) \leq d(F)$. If $i = j$ and $p \geq t_i$ and $q \geq t_i$, then $d((p, i), (q, j)) \leq m$. The result thus follows. □

Let P_n be the path of order n.

Theorem 1 *$T_i(s_1, s_2, \ldots, s_n) \in \mathcal{C}_2 \cup \mathcal{C}_1$, where $i = 3, 4$.*

Proof. It was shown in [18] that $P_4(s_1, s_2, s_3, s_4) \in \mathcal{C}_0 \cup \mathcal{C}_1$. Note that $T_3(s_1, s_2, \ldots, s_n) \cong P_4(\sum_{n(i_j)=1} s_j, s_2, s_3, \sum_{n(i_j)=4} s_j)$. The result follows for $i = 3$.

We shall now prove the result for $i = 4$. Define an orientation F of $T_4^{(2)}$ as follows: for $n(i) = 2$ or 4, $(1, i) \to (1, 3) \to (2, i) \to (2, 3) \to (1, i)$ and $(2, i) \to \{(1, j), (2, j)\} \to (1, i)$, where j is adjacent to i in T_4 and $n(j) = \begin{cases} 1 & \text{if} \quad i = 2 \\ 5 & \text{if} \quad i = 4 \end{cases}$.
As an illustration, the orientation F of a $T_4^{(2)}$ is shown in Figure 2.

Observe the following facts about F:

(i) for $n(i) = 1, 5$ and $p = 1, 2$, $d((p, i), (1, 3)) = d((1, 3), (p, i)) = 2$;

(ii) for $n(i) = 1, 5$ and $p = 1, 2$, $d((p, i), (2, 3)) = d((2, 3), (p, i)) = 4$;

(iii) for $n(i) = 2, 4$, $d((1, i), (1, 3)) = 1$, $d((1, i), (2, 3)) = 3$, $d((1, 3), (1, i)) = 3$ and $d((2, 3), (1, i)) = 1$;

(iv) for $n(i) = 2, 4$, $d((2, i), (1, 3)) = 3$, $d((2, i), (2, 3)) = 1$, $d((1, 3), (2, i)) = 1$ and $d((2, 3), (2, i)) = 3$;

(v) $d((1, 3), (2, 3)) = d((2, 3), (1, 3)) = 2$.

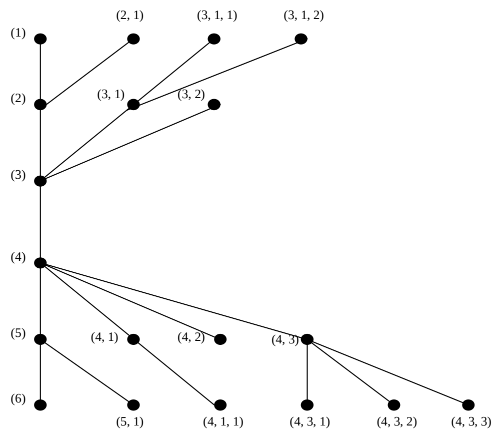

Figure 2

We shall now show that $d(F)=5$ by showing that for all $u\in V(F)$, $e(u)\le 5$. From observations (i)-(v), $e((1,3))=3$ and $e((2,3))=4$. For $n(i)=1,5$ and $p=1,2, e((p,i))\le d((p,i),(1,3))+e((1,3))=2+3=5$. For $n(i)=2,4, e((1,i))\le d((1,i),\ (1,3))+e((1,3))=1+3=4$ and $e((2,i))\le d((2,i),(2,3))+e((2,3))=1+4=5$. All cases have been covered and so $d(F)=5$. Since every vertex in F lies on a cycle of length 4, we have $\vec{d}\,(T_4(s_1,s_2,\ldots,s_n))\le 5$ by Lemma 1. Thus $T_4(s_1,s_2,\ldots,s_n)\in\mathcal{C}_0\cup\mathcal{C}_1$. $\Box$ To prove the next theorem in this section, we shall need the following lemma.

Lemma 2 *There is exactly one orientation of $P_5^{(2)}$ with diameter 4 up to isomorphism.*

Proof. Suppose there exists an $F\in\mathcal{D}(P_5^{(2)})$ such that $d(F)=4$. Since F is strong, we may assume that $(2,2)\to(1,1)\to(1,2)$ and $(1,4)\to(1,5)\to(2,4)$. Since $d((1,1),(1,5))\le 4$, there must be a $(1,2)$-$(1,4)$ path of length 2. We may assume that $(1,2)\to(1,3)\to(1,4)$. Since $d((1,4),(1,1))\le 4$, $(1,4)\to(2,3)\to(2,2)$. Since $d((1,5),(1,2))\le 4$, $(2,4)\to(2,3)\to(1,2)$. Since $d((1,1),(2,4))\le 4$, $(1,3)\to(2,4)$. Since $d((1,5),(2,5))\le 4$ and F is strong, $(2,4)\to(2,5)\to(1,4)$. Since $d((2,2),(1,5))\le 4$, $(2,2)\to(1,3)$. Since $d((1,1),(2,1))\le 4$ and F is strong, $(1,2)\to(2,1)\to(2,2)$. Thus F is isomorphic to the orientation X^1 of Figure 3, which is of diameter 4. $\Box$

Corollary Let T_4 be a tree of diameter 4 which contains $P_5: i_1i_2\ldots i_5$ as a subgraph such that $\deg_{T_4}(i_1)=\deg_{T_4}(i_5)=1$, $\deg_{T_4}(i_2)=\deg_{T_4}(i_4)=2$ and $\deg_{T_4}(i_3)\ge 2$. If there exists $F\in\mathcal{D}(T_4^{(2)})$ such that $d(F)=4$, then $F[V(P_5^{(2)})]\cong X^1$.

X^1 :

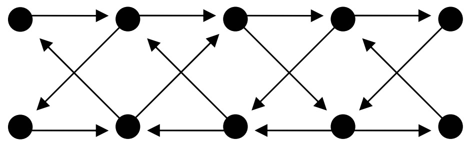

Figure 3

X^1 :

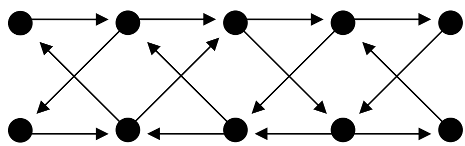

Figure 4

Proof. Observe that the proof of Lemma 2 is independent of whether there exist any new edges incident with (1, 3) or (2, 3). The result thus follows. □

Call $F \in \mathcal{D}(P_5^{(2)})$ a **symmetric** orientation if there exists an isomorphism φ : $F_1 \cup F_2 \cup F_3 \to F_5 \cup F_4 \cup F_3$ such that for $p = 1, 2$, $\varphi((p, i)) = (p, 6 - i)$.

Consider the orientation X^1 of Figure 3. We observe that

(i) X^1 is not symmetric;

(ii) $s_{X_2^1 \cup X_3^1}((1, 3)) = s_{X_3^1 \cup X_4^1}((2, 3)) = 0$.

Theorem 2

(I) If $\deg((3)) = 2$, *then* $T_4(s_1, s_2, \ldots, s_n) \in \mathcal{C}_0$.

(II) If $\deg((3)) \geq 3$, *then* $T_4^{(2)} \in \mathcal{C}_1$.

Proof. We shall first prove (I). It is easy to see that every vertex in X^1 lies on a cycle of length 4. Hence by Lemma 1, $\vec{d}(P_5(s_1, s_2, s_3, s_4, s_5)) = 4$. Let T_4 be a tree of diameter 4 such that $\deg((3)) = 2$. Note that $T_4(s_1, s_2, \ldots, s_n) \cong P_5(\sum_{n(i_j)=1} s_j, \; s_2, s_3, s_4, \sum_{n(i_j)=5} s_j)$. Thus we have $\vec{d}(T_4(s_1, s_2, \ldots, s_n)) = 4$ and $T_4(s_1, s_2, \ldots, s_n) \in \mathcal{C}_0$.

We shall now prove (II). Let T_4 be a tree of diameter 4 such that $\deg((3)) \geq 3$.

Assume that there exist at least 3 vertices in T_4 such that the distance between any two of them is 4. We need only consider T of Figure 4.

Suppose there exists $H \in \mathcal{D}(T^{(2)})$ such that $d(H) = 4$. By the corollary to Lemma 2, $H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5 \cong H_1 \cup H_2 \cup H_3 \cup H_{(3,1)} \cup H_{(3,1,1)} \cong H_5 \cup H_4 \cup$

X^1 :

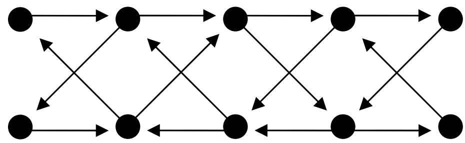

Figure 5

$H_3 \cup H_{(3,1)} \cup H_{(3,1,1)} \cong X^1$. But this is impossible since X^1 is not symmetric by the above observation (i).

Assume now that there exist exactly 2 vertices in T_4 such that the distance between them is 4. We need only consider T of Figure 5.

Suppose there exists $H \in \mathcal{D}(T^{(2)})$ such that $d(H) = 4$. By the corollary to Lemma 2, $H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5 \cong X^1$. By the above observation (ii), we have $s_{H_2 \cup H_3}((1,3)) = s_{H_2 \cup H_3}((2,3)) = 0$. Since H is strong, $s((1,(3,1))) = 1$. If $(1,(3,1)) \to (1,3)$, then $d((1,(3,1)),(1,1)) = d((1,(3,1)),(2,3)) + d((2,3),(1,1)) = 3 + 2 = 5$, a contradiction. If $(1,(3,1)) \to (2,3)$, then $d((1,(3,1)),(1,5)) = d((1,(3,1)),(1,3)) + d((1,3),(1,5)) = 3 + 2 = 5$, a contradiction again.

Hence $\vec{d}(T_4^{(2)}) \geq 5$. By Theorem 1, $\vec{d}(T_4^{(2)}) = 5$ and result (II) follows. □

4 Optimal orientations of $T_d(s_1, s_2, \ldots, s_n)$, where $d \geq 5$

In this section, we shall turn our attention to $T_d(s_1, s_2, \ldots, s_n)$, where $d \geq 5$. We shall divide our consideration into 2 cases, i.e., T_5 and T_d with $d \geq 6$.

We shall need a few preliminary results on orientations of $P_5^{(2)}$ and $P_6^{(2)}$.

Lemma 3 *There are exactly 3 non-isomorphic orientations of $P_6^{(2)}$ with diameter 5.*

Proof. Suppose there exists an $F \in \mathcal{D}(P_6^{(2)})$ such that $d(F) = 5$. We shall split our argument into 2 cases by considering the orientation of $F_5 \cup F_6$.

<u>Case 1</u> $(1,5) \to (1,6) \to (2,5) \to (2,6) \to (1,5)$.

Let $u \in V(F_1)$, $v \in V(F_2)$ and $w \in V(F_5)$. We have the following observations:

(1a) since $\bar{s}_F((1,6)) = \bar{s}_F((2,6)) = 1$, $(1,5) \to (1,6)$ and $(2,5) \to (2,6)$, we have $d(u,w) = 4$ and $d(v,w) = 3$;

(1b) since $s_F((1,6)) = s_F((2,6)) = 1$, $(1,6) \to (2,5)$ and $(2,6) \to (1,5)$, we have $d(w,u) = 4$ and $d(w,v) = 3$.

Since F is strong, we may assume that $(2,2) \to (1,1) \to (1,2)$. Since $d((1,1),(1,5)) = 4$ by observation (1a), there must be a (1, 2)-(1, 5) path of length 3. We may assume that $(1,2) \to (1,3) \to (1,4) \to (1,5)$. By observation (1b), $d((1,5),(1,2)) = 3$ and thus $(1,5) \to (2,4) \to (2,3) \to (1,2)$. We shall further divide our consideration into 2 subcases.

X^1 :

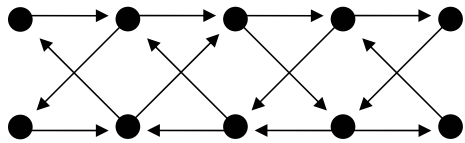

Figure 6

X^1 :

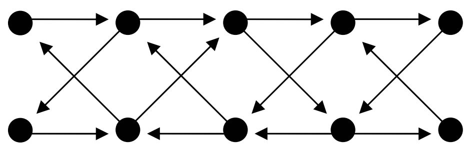

Figure 7

1.1. $(2,2) \to (2,3)$.
By observation (1b), $d((1,5),(2,2)) = 3$ and thus $(2,4) \to (1,3) \to (2,2)$. By observation (1a), $d((2,2),(2,5)) = 3$ and thus $(2,3) \to (1,4) \to (2,5)$. By observation (1b), $d((2,5),(1,2)) = 3$ and thus $(2,5) \to (2,4)$. Now either $(1,2) \to (2,1) \to (2,2)$ or $(2,2) \to (2,1) \to (1,2)$. This gives rise to 2 orientations Y^1 and Y^2 with $d(Y^1) = d(Y^2) = 5$ as shown in Figure 6.
1.2. $(2,3) \to (2,2)$.
By observation (1a), $d((2,2),(1,5)) = 3$ and thus $(2,2) \to (1,3)$. Since $d((1,1),(2,1)) \leq 5$, we must have $(1,2) \to (2,1)$ and this in turn leads to $(2,1) \to (2,2)$ since F is strong. Suppose $(2,5) \to (1,4)$. By observation (1a), $d((2,2),(2,5)) = 3$ and thus $(1,3) \to (2,4) \to (2,5)$. By observation (1b), $d((2,5),(2,2)) = 3$ and thus $(1,4) \to (2,3)$. This gives rise to an orientation which is isomorphic to Y^1. Now suppose $(1,4) \to (2,5)$. By observation (1b), $d((2,5),(1,1)) = 4$ and thus $(2,5) \to (2,4)$. At this stage, we have a partial orientation Z of $P_6^{(2)}$. This partial orientation Z gives rise to 2 non-isomorphic orientations Y^3 and Y^4 as shown in Figure. It is easy to check that $d(Y^3) = 5$ and $d(Y^4) = 6$ (since $d((1,1),(2,3)) = 6$).
<u>Case 2</u> $(1,5) \to \{(1,6),(2,6)\} \to (2,5)$.

Let $u \in V(F_1)$ and $v \in V(F_2)$. We have the following observations:

(2a) since $\bar{s}_F((1,6)) = \bar{s}_F((2,6)) = 1$ and $(1,5) \to \{(1,6),(2,6)\}$, we have $d(u,(1,5)) = 4$ and $d(v,(1,5)) = 3$;

(2b) since $s_F((1,6)) = s_F((2,6)) = 1$ and $\{(1,6),(2,6)\} \to (2,5)$, we have $d((2,5),u) = 4$ and $d((2,5),v) = 3$.

Since F is strong, we may assume that $(2,2) \to (1,1) \to (1,2)$. Since $d((1,1),(1,5)) = 4$ by observation (2a), there must be a (1, 2)-(1, 5) path of length 3. We may

X^1 :

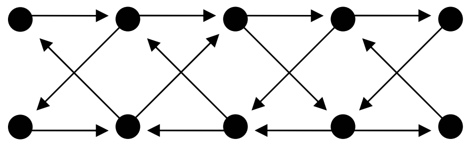

Figure 8

assume that $(1,2) \to (1,3) \to (1,4) \to (1,5)$. By observation (2b), $d((2,5),(1,2)) = 3$ and thus $(2,3) \to (1,2)$. Since $d((1,5),(1,1)) \le 5$, we have $(1,5) \to (2,4)$. Since $d((1,6),(2,6)) \le 5$, we must have $(2,5) \to (1,4)$. Since $d((1,1),(2,5)) \le 5$, we have $(1,3) \to (2,4) \to (2,5)$. By observation (2b), $d((2,5),(1,1)) = 4$ and thus $(1,4) \to (2,3) \to (2,2)$. By observation (2a), $d((2,2),(1,5)) = 3$ and thus $(2,2) \to (1,3)$. Since $d((1,5),(1,1)) \le 5$, we have $(2,4) \to (2,3)$. Since $d((1,1),(2,1)) \le 5$ and F is strong, we have $(1,2) \to (2,1) \to (2,2)$. This will result in orientation Y^5 as shown in Figure 8. However, note that $Y^5 \cong Y^2$.

We have considered all possible cases and obtained exactly 3 non-isomorphic orientations of diameter 5, i.e., Y^1, Y^2 and Y^3. □

Corollary Let T_5 be a tree of diameter 5 which contains $P_6 : i_1 i_2 \cdots i_6$ as a subgraph such that $\deg_{T_5}(i_1) = \deg_{T_5}(i_6) = 1$, $\deg_{T_5}(i_2) = \deg_{T_5}(i_5) = 2$, $\deg_{T_5}(i_3) \ge 2$ and $\deg_{T_5}(i_4) \ge 2$. If there exists $F \in \mathcal{D}(T_5^{(2)})$ such that $d(F) = 5$, then $F[V(P_6^{(2)})]$ is isomorphic to one of Y^1, Y^2, Y^3 or Y^4.

Proof. Observe that the proof of Lemma 3, up to the partial orientation Z in Case 1.2 and in its entirety for the other cases, is independent of whether there exist any new edges incident with (p,i), where $p = 1,2$ and $i = 3,4$. The result thus follows. □

Lemma 4

(I) If $F \in \mathcal{D}(P_5^{(2)})$ is symmetrical, then $d(F) \ge 5$.

(II) There exists exactly one symmetrical orientation $F \in \mathcal{D}(P_5^{(2)})$, up to isomorphism, such that $d(F) = 5$.

Proof. By Lemma 2, X^1 is the only orientation of $P_5^{(2)}$, up to isomorphism, with diameter 4. By the observation (i) following the corollary to Lemma 2, X^1 is not symmetric. Result (I) follows.

Suppose there exists a symmetrical orientation $F \in \mathcal{D}(P_5^{(2)})$ such that $d(F) = 5$. Since F is strong, we may assume that $(2,2) \to (1,1) \to (1,2)$ and by symmetry, $(2,4) \to (1,5) \to (1,4)$. Since $d((1,1),(1,5)) \le 5$, there must be a (1, 2)-(2, 4) path of length 2. We may assume that $(1,2) \to (1,3) \to (2,4)$ and by symmetry, $(1,4) \to (1,3) \to (2,2)$. Suppose $(1,2) \to (2,1)$. Then since F is strong, $(2,1) \to (2,2)$; and by symmetry, $(1,4) \to (2,5) \to (2,4)$. Since $d((1,1),(2,5)) \le 5$, $(1,2) \to (2,3) \to$

X^1 :

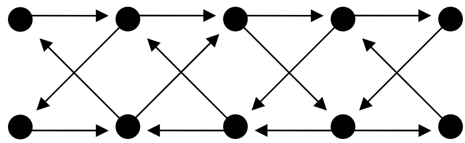

Figure 9

$(1,4)$, a contradiction to the symmetry of F. Thus $(2,2) \to (2,1) \to (1,2)$ and by symmetry, $(2,4) \to (2,5) \to (1,4)$. Suppose $(2,3) \to (2,4)$. Since $d((2,4),(1,2)) \leq 5$, $(1,4) \to (2,3) \to (1,2)$, a contradiction to the symmetry of F. Thus $(2,4) \to (2,3)$ and by symmetry, $(2,2) \to (2,3)$. If $(1,4) \to (2,3)$, then by symmetry, $(1,2) \to (2,3)$ and so $d((2,2),(1,4)) = 6$, a contradiction. Hence $(1,2) \leftarrow (2,3) \to (1,4)$ and so F must be isomorphic to the orientation X^2 of $P_5^{(2)}$ as shown in Figure 9. $\square$

Corollary Let T_5 be a tree of diameter 5 which contains $P_5 : i_1 i_2 \ldots i_5$ as a subgraph such that $\deg_{T_5}(i_1) = \deg_{T_5}(i_5) = 1$, $\deg_{T_5}(i_2) = \deg_{T_5}(i_4) = 2$ and $\deg_{T_5}(i_3) \geq 2$. If there exists $F \in \mathcal{D}(T_5^{(2)})$ such that $d(F) = 5$ and $F[V(P_5^{(2)})]$ is symmetric, then $F[V(P_5^{(2)})] \cong X^2$.

Proof. Observe that the proof of Lemma 4(II) is independent of whether there exist any new edges incident with (1, 3) or (2, 3). The result thus follows. $\square$

Remark Note that $s_{X_1^2 \cup X_2^2}((1,2)) = 0$. Thus, if $F \in \mathcal{D}(P_6^{(2)})$ with $d(F) = 5$ is such that there is an isomorphism $\varphi : F_1 \cup F_2 \cup F_3 \to X_1^2 \cup X_2^2 \cup X_3^2$ with $\varphi((p,i)) = (p,i)$, then $F \cong Y^2$.

We are now ready to establish the following main result for T_5.

Let $A = \{x \in V(T_5) | d(x,u) = 5 = d(x,v) \text{ for some } u,v \in V(T_5), u \neq v\}$.

Theorem 3 *Let T_5 be a tree of diameter 5. Then*

(I) $T_5(s_1, s_2, \ldots, s_n) \in \mathcal{C}_0 \cup \mathcal{C}_1$*;*

(II) if $|A| \leq 1$*, then* $T_5(s_1, s_2, \ldots, s_n) \in \mathcal{C}_0$*;*

(III) if $\deg(v) \leq 2$ *for all* $v \notin \{(1,3),(2,3),(1,4),(2,4)\}$ *and* $|A| \geq 2$*, then* $T_5^{(2)} \in \mathcal{C}_1$*.*

Proof. We shall prove (I) by defining an orientation (suggested by the remark above) $F \in \mathcal{D}(T_5^{(2)})$ such that $d(F) \leq 6$ as follows:

$$(p,i) \to (q,j) \text{ if and only if } (p,n(i)) \to (q,n(j)) \text{ in } Y^2.$$

As an illustration, the orientation F of a $T_5^{(2)}$ with some vertices labeled is shown in Figure 10.

X^1 :

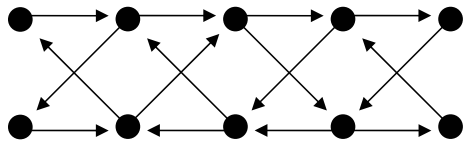

Figure 10

Let u and v be any 2 vertices in F. Observe that the shortest path between u and v lies on a digraph isomorphic to one of the following:

$$\begin{aligned}
F^a &= F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5 \cup F_6;\\
F^b &= F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_{(4,2)};\\
F^c &= F_1 \cup F_2 \cup F_3 \cup F_{(3,2)};\\
F^d &= F_1 \cup F_2 \cup F_3 \cup F_{(3,1)} \cup F_{(3,1,1)};\\
F^e &= F_1 \cup F_2 \cup F_3 \cup F_{(2,1)};\\
F^f &= F_{(3,2)} \cup F_3 \cup F_4 \cup F_5 \cup F_6;\\
F^g &= F_{(3,2)} \cup F_3 \cup F_4 \cup F_{(4,2)};\\
F^h &= F_{(3,2)} \cup F_3 \cup F_{(3,3)};\\
F^i &= F_{(4,2)} \cup F_3 \cup F_4 \cup F_5 \cup F_6;\\
F^j &= F_{(4,2)} \cup F_4 \cup F_3 \cup F_{(4,3)};\\
F^k &= F_{(4,1,1)} \cup F_{(4,1)} \cup F_4 \cup F_3 \cup F_5 \cup F_6.
\end{aligned}$$

It can be checked that F^a to F^j have diameter not exceeding 5 (note that $F^a \cong Y^2$ and $F^d \cong X^2$). It can also be checked that F^k has diameter 6. Hence $d(F) \le 6$ and thus, by Lemma 1, we have result (I). Suppose $|A| = 0$. Then no subdigraph of F will be isomorphic to F^k. If $|A| = 1$, we may let $(6) \in A$. Again, no subdigraph of F will be isomorphic to F^k. Thus, $d(F) = 5$ and by Lemma 1, we have result (II).

Suppose $|A| \ge 2$. Label two of these vertices (6) and (4, 1, 1). Then $d_F((1,(6)), (1,(4,1,1))) = 6$ and thus $d(F) = 6$. Hence, for such a tree T_5, to show that $\vec{d}(T_5^{(2)}) = 5$, we need to introduce an orientation of T_5 different from F. We need only consider T of Figure 11.

Suppose there exists $H \in \mathcal{D}(T^{(2)})$ such that $d(H) = 5$. Let

$$\begin{aligned}
H^1 &= H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5 \cup H_6,\\
H^2 &= H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_{(4,1)} \cup H_{(4,1,1)},\\
H^3 &= H_{(3,1,1)} \cup H_{(3,1)} \cup H_3 \cup H_4 \cup H_5 \cup H_6 \quad \text{and}\\
H^4 &= H_{(3,1,1)} \cup H_{(3,1)} \cup H_3 \cup H_4 \cup H_{(4,1)} \cup H_{(4,1,1)}.
\end{aligned}$$

Then by the corollary to Lemma 3, each of the H^i, $i = 1, 2, 3, 4$, must be isomorphic to one of Y^1, Y^2, Y^3, Y^4.

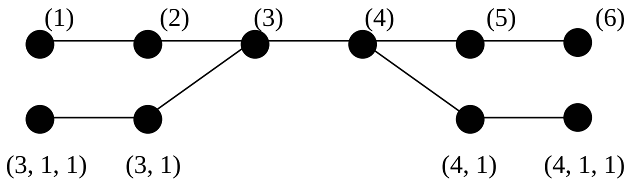

Figure 11

Suppose there exists an isomorphism $\varphi : H^1 \to Y^3$. By symmetry, we may assume that $\varphi((p,i)) = (p,i)$ for all p and all i. Observe that $Y_3^3 \cup Y_4^3$ is not isomorphic to any of $Y_3^1 \cup Y_4^1$, $Y_3^2 \cup Y_4^2$, $Y_3^4 \cup Y_4^4$. Thus, each of the H^i, $i = 2, 3, 4$, must also be isomorphic to Y^3. Hence $H_1 \cup H_2 \cup H_3 \cup H_{(3,1)} \cup H_{(3,1,1)}$ is symmetric but not isomorphic to X^2, a contradiction to the corollary to Lemma 4. Thus, by symmetry, Y^3 is not isomorphic to any of H^i, $i = 1, 2, 3, 4$.

Next, suppose there exists an isomorphism $\varphi : H^1 \to Y^2$. By symmetry, we may assume that $\varphi((p,i)) = (p,i)$ for all p and all i. Suppose $H^2 \cong Y^2$. Thus $H_6 \cup H_5 \cup H_4 \cup H_{(4,1)} \cup H_{(4,1,1)}$ is symmetric but not isomorphic to X^2, a contradiction to the corollary to Lemma 4. Now H^2 cannot be isomorphic to Y^1 because $Y_1^2 \cup Y_2^2$ is neither isomorphic to $Y_1^1 \cup Y_2^1$ nor to $Y_5^1 \cup Y_6^1$. The same argument can be used to show that H^2 cannot be isomorphic to Y^4. Hence $d(H^2) \geq 6$, a contradiction. Thus, by symmetry, Y^2 is not isomorphic to any of H^i, $i = 1, 2, 3, 4$.

Now suppose there exists an isomorphism $\varphi : H^1 \to Y^1$. By symmetry, we may assume that $\varphi((p,i)) = (p,i)$ for all p and all i. Suppose $H^2 \cong Y^1$. Then, $H_6 \cup H_5 \cup H_4 \cup H_{(4,1)} \cup H_{(4,1,1)}$ is symmetric but not isomorphic to X^2, a contradiction to the corollary to Lemma 4. Thus $H^2 \cong Y^4$. It follows that $H^3 \cong Y^1$. Then, $H_1 \cup H_2 \cup H_3 \cup H_{(3,1)} \cup H_{(3,1,1)}$ is symmetric but not isomorphic to X^2, a contradiction to the corollary to Lemma 4. Thus, by symmetry, Y^1 is not isomorphic to any of H^i, $i = 1, 2, 3, 4$.

Finally, we must have $Y^4 \cong H^i$, $i = 1, 2, 3, 4$. Hence $H_1 \cup H_2 \cup H_3 \cup H_{(3,1)} \cup H_{(3,1,1)}$ is symmetric but not isomorphic to X^2, a contradiction to the corollary to Lemma 4.

Hence $d(H) \geq 6$ and result (III) follows from result (I). □

Finally, we shall consider trees of diameter at least 6. In Theorem 2, it was shown that if $d(G) \geq 4$ and $s_i \geq 4$ for each $i = 1, 2, \ldots, n$, then $G(s_1, s_2, \ldots, s_n) \in \mathcal{C}_0$. Theorem 4 below extends this result to include the case that $2 \leq s_i \leq 3$ when G is a tree of diameter at least 6.

Theorem 4 *Let T_d be a tree of order n and diameter d, where $d \geq 6$. Then $T_d(s_1, s_2, \ldots, s_n) \in \mathcal{C}_0$.*

Proof. We shall consider 2 cases according to the parity of d.

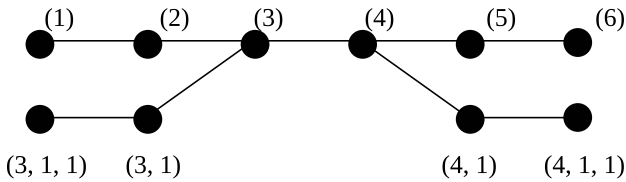

Figure 12

Case 1 $d \equiv 0 (\mathrm{mod}\ 2)$.
We shall first design an orientation for $T_4^{(2)}$, and based on this, we shall then design optimal orientations for $T_d^{(2)}$, $d \geq 6$. Let $S = \{i \in V(T_4) | \deg(i) = 1 \text{ and } n(i) = 4\}$.

Define an orientation F of $T_4^{(2)}$ as follows:

(i) for $i_4 \notin S$, $\{(1, i_1), (2, i_1)\} \to (1, 2) \to \{(1, 3), (2, 3)\} \to (2, i_4) \to \{(1, i_5), (2, i_5)\} \to (1, i_4) \to \{(1, 3), (2, 3)\} \to (2, 2) \to \{(1, i_1), (2, i_1)\}$, where $n(i_v) = v$;

(ii) for $i \in S$, $(1, i) \to (1, 3) \to (2, i) \to (2, 3) \to (1, i)$.

As an illustration, the orientation F of a $T_4^{(2)}$ is shown in Figure 12.

Observe the following facts about F:

(1a) for $u = (p, i)$, where $p = 1, 2$, $n(i) = 2, 4$ and $i \notin S$,
$d(u, (1, 3)) = d(u, (2, 3)) \leq 3$ and $d((1, 3), u) = d((2, 3), u) \leq 3$;

(1b) for $u = (p, i)$, where $p = 1, 2$ and $n(i) = 1, 5$,
$d(u, (1, 3)) = d(u, (2, 3)) = d((1, 3), u) = d((2, 3), u) = 2$;

We shall prove that $d(F) = 6$ by showing that $e(u) \leq 6$ for all $u \in V(F)$. We shall consider 4 subcases.

1. $u \in V(F_3)$.
By symmetry, we need only consider $u = (1, 3)$. By observations (1a), (1b) and the fact that $(1, 3) \to (2, i) \to (2, 3) \to (1, i)$ for $i \in S$, we have $e(u) = 3$ if there exists $i \in S$. Otherwise, $e(u) = 4$ since $\{(1, 3), (2, 3)\} \to (2, 2) \to (1, 1) \to (1, 2) \to \{(1, 3), (2, 3)\}$.

2. $u \in V(F_i)$, where $i \in S$.
By symmetry, we need only consider $u = (1, i)$.

2_1. $v \in V(F3)$.
$(1, i)(1, 3)(2, i)(2, 3)$ is a path of length 3.

2_2. $v \in V(F_j)$, where $j \in S$.
$(1, i)(1, 3)(2, j)(2, 3)(1, j)$ is a path of length 4.

2_3. $v \in V(F \backslash (F3 \cup \bigcup_{i \in S} F_i))$.
By (1), $d((1,3),v) \leq 3$. Since $(1,i) \to (1,3)$, we have $d(u,v) \leq d(u,(1,3)) + d((1,3),v) \leq 1+3=4$.

3. $u \in V(F_i)$, where $n(i) = 2,4$ and $i \notin S$.
From (1), $d((1,3),v) \leq 3$ for $v \neq (2,3)$. By observation (1a), $d(u,(1,3)) \leq 3$. Thus, $d(u,v) \leq d(u,(1,3)) + d((1,3),v) \leq 3+3=6$ for $v \neq (2,3)$. However, by observation (1a) again, $d(u,(2,3)) \leq 3$. Hence $e(u) \leq 6$.

4. $u \in V(F_i)$, where $n(i) = 1,5$.
By observation (1b), $d(u,(1,3)) = 2$. By (1), $d((1,3),v) \leq 3$ for $v \neq (2,3)$. Thus, $d(u,v) \leq d(u,(1,3)) + d((1,3),v) \leq 2+3=5$ for $v \neq (2,3)$. However, by observation (1b) again, $d(u,(2,3)) = 2$. Hence $e(u) \leq 5$.

We have covered all possible cases. Note that $d(u,v) = 6$ if and only if $u = (2,i)$ and $v = (1,j)$ for distinct i,j, where $(n(i),n(j)) = (2,4),(4,2)$ or $(4,4)$, and $i \neq S$. Thus $d(F) = 6$.

Now let T_d be a tree of diameter d, $d \geq 6$. Denote by $T^{(2)}$ the induced subgraph of $T_d^{(2)}$, where $V(T^{(2)}) = \{(p,i) | p = 1,2 \text{ and } \frac{d}{2} - 1 \leq n(i) \leq \frac{d}{2} + 3\}$. Let $F \in \mathcal{D}(T_4^{(2)})$, where $T_4^{(2)} \cong T^{(2)}$, be as defined above. Define $H \in \mathcal{D}(T_d^{(2)})$ as follows:

(i) $\varphi : F \to H[V(T^{(2)})]$ is an isomorphism such that $\varphi(v) = u$ iff $v^i = u^i$ when $i \neq 1$ and $v^1 = u^1 + \frac{d}{2} - 2$;

(ii) for all other edges, $(1,i) \to (1,j) \to (2,i) \to (2,j) \to (1,i)$ iff $n(i) < n(j)$.

For each $u \notin V(T^{(2)})$, let u' be the vertex in $T^{(2)}$ of minimum distance from u and let u'' be the vertex in $T^{(2)}$ of minimum distance to u. Note that $n(u') = \frac{d}{2} - 1$ or $\frac{d}{2} + 3$ and $n(u'') = \frac{d}{2} - 1$ or $\frac{d}{2} + 3$. Observe also the following facts about H:

(2a) for $u \notin V(T^{(2)})$, $d(u,u') \leq \frac{d-4}{2}$ and $d(u'',u) \leq \frac{d-4}{2}$;

(2b) for $u,v \notin V(T^{(2)})$, $d(u',v'') = 4$ (by observation (1b)).

Let $u_1,u_2 \notin V(T^{(2)})$ and $v_1,v_2 \in V(T^{(2)})$. By observations (2a) and (2b) above, $d(u_1,u_2) \leq \frac{d-4}{2} + 4 + \frac{d-4}{2} = d$, $d(u_1,v_1) \leq \frac{d-4}{2} + 5 \leq d$ and $d(v_1,u_1) \leq 5 + \frac{d-4}{2} \leq d$. In addition, since $d(F) = 6$, $d(v_1,v_2) \leq 6 \leq d$. Hence $d(H) = d$. Since every vertex in H lies on a cycle of length 4, by Lemma 1, we have the result for $d \equiv 0 (\text{mod } 2)$.

Case 2 $d \equiv 1 (\text{mod } 2)$.
We shall first design an orientation for $T_5^{(2)}$, and based on this, we shall then design optimal orientations for $T_d^{(2)}$, $d \geq$ Let $S = \{i \in V(T_4) | \deg(i) = 1 \text{ and } n(i) = 2,5\}$.

Define an orientation F of $T_5^{(2)}$ as follows:

(i) for $i_2, i_5 \notin S$, $\{(1,i_1),(2,i_1)\} \to (1,i_2) \to \{(1,3),(2,3)\} \to (2,i_2) \to \{(1,i_1),(2,i_1)\}$, $\{(1,i_4),(2,i_4)\} \to (2,i_5) \to \{(1,i_6),(2,i_6)\} \to (1,i_5) \to \{(1,i_4),(2,i_4)\}$ and $(1,3) \to (1,4) \to (2,3) \to (2,4) \to (1,3)$, where $n(i_v) = v$;

(ii) for $i \in S$, $(1,i) \to (1,j) \to (2,i) \to (2,j) \to (1,i)$, where $ij \in E(T_5)$.

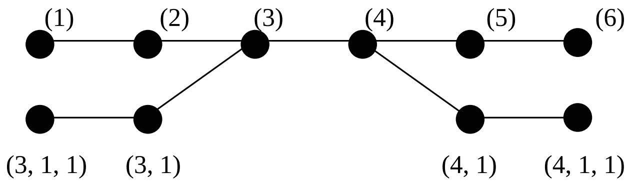

Figure 13

As an illustration, the orientation F of a $T_5^{(2)}$ is shown in Figure 13.

Observe the following facts about F:

(3a) for $u = (p, i)$, where $p = 1, 2$, $n(i) = 2$ and $i \notin S$,
$d(u, (1, 3)) = d(u, (2, 3)) \leq 3$, $d(u, (1, 4)) = d(u, (2, 4)) \leq 4$, $d((1, 3), u) = d((2, 3), u) \leq 3$ and $d((1, 4), u) = d((2, 4), u) \leq 4$;

(3b) for $u = (p, i)$, where $p = 1, 2$, $n(i) = 5$ and $i \notin S$,
$d(u, (1, 4)) = d(u, (2, 4)) \leq 3$, $d(u, (1, 3)) = d(u, (2, 3)) \leq 4$, $d((1, 4), u) = d((2, 4), u) \leq 3$ and $d((1, 3), u) = d((2, 3), u) \leq 4$;

(3c) for $u = (p, i)$, where $p = 1, 2$ and $n(i) = 1$,
$d(u, (1, 3)) = d(u, (2, 3)) = d((1, 3), u) = d((2, 3), u) = 2$ and $d(u, (1, 4)) = d(u, (2, 4)) = d((1, 4), u) = d((2, 4), u) = 3$;

(3d) for $u = (p, i)$, where $p = 1, 2$ and $n(i) = 6$,
$d(u, (1, 4)) = d(u, (2, 4)) = d((1, 4), u) = d((2, 4), u) = 2$ and $d(u, (1, 3)) = d(u, (2, 3)) = d((1, 3), u) = d((2, 3), u) = 3$.

We shall prove that $d(F) = 7$ by showing that $e(u) \leq 7$ for all $u \in V(F)$. We shall consider 4 subcases. (The proof follows closely the proof of Case 1.)

1. $u \in V(F_3) \cup V(F_4)$.
By symmetry, we need only consider $u = (1, 3)$. By observations (3a)-(3d) and the facts that $(1, 3) \to (2, i) \to (2, 3) \to (1, i)$ or $(1, 3) \to (1, 4) \to (2, i) \to (2, 4) \to (1, i)$ for $i \in S$ and that $(1, 3) \to (1, 4) \to (2, 3) \to (2, 4)$, we have $e(u) = 4$.

2. $u \in V(F_i)$, where $i \in S$.
By symmetry, we need only consider $u = (1, i)$, where $n(i) = 2$.

2_1. $v \in V(F_3) \cup V(F_4)$.
$(1, i)(1, 3)(1, 4)(2, 3)(2, 4)$ is a path of length 4.

2_2. $v \in V(F_j)$, where $j \in S$.
If $n(j) = 2$, then $(1, i)(1, 3)(2, j)(2, 3)(1, j)$ is a path of length 4.
If $n(j) = 4$, then $(1, i)(1, 3)(1, 4)(2, j)(2, 4)(1, j)$ is a path of length 5.

2_3. $v \in V(F \backslash (F_3 \cup F_4 \cup \bigcup_{i \in S} F_i))$.
By observations (3a)-(3d), $d((1,3),v) \leq 4$. Since $(1,i) \to (1,3)$, we have $d(u,v) \leq d(u,(1,3)) + d((1,3),v) \leq 1+4 = 5$.

3. $u \in V(F_i)$, where $n(i) = 2,5$ and $i \notin S$.
By symmetry, we need only consider the case when $n(i) = 2$. From (1), $d((1,3),v) \leq 4$. By observation (3a), $d(u,(1,3)) \leq 3$. Thus, $d(u,v) \leq d(u,(1,3)) + d((1,3),v) \leq 3+4 = 7$. Hence $e(u) \leq 7$.

4. $u \in V(F_i)$, where $n(i) = 1,6$.
By symmetry, we need only consider $u = (1,1)$. By observation (1b), $d(u,(1,3)) = 2$. By (1), $d((1,3),v) \leq 4$. Thus, $d(u,v) \leq d(u,(1,3)) + d((1,3),v) \leq 2+4 = 6$. Hence $e(u) \leq 6$.

We have covered all possible cases. Note that $d(u,v) = 7$ if and only if $u = (2,i)$ and $v = (1,j)$ for distinct i,j, where $(n(i),n(j)) = (2,5)$ or $(5,2)$, and $i \notin S$. Thus $d(F) = 7$.

Now let T_d be a tree of diameter d, $d \geq$ Denote by $T^{(2)}$ the induced subgraph of $T_d^{(2)}$, where $V(T^{(2)}) = \{(p,i) | p = 1,2 \text{ and } \frac{d-3}{2} \leq n(i) \leq \frac{d+7}{2}\}$. Let $F \in \mathcal{D}(T_5^{(2)})$, where $T_5^{(2)} \cong T^{(2)}$, be as defined above. Define $H \in \mathcal{D}(T_d^{(2)})$ as follows:

(i) $\varphi : F \to H[V(T^{(2)})]$ is an isomorphism such that $\varphi(v) = u$ iff $v^i = u^i$ when $i \neq 1$ and $v^1 = u^1 + \frac{d-5}{2}$;

(ii) for all other edges, $(1,i) \to (1,j) \to (2,i) \to (2,j) \to (1,i)$ iff $n(i) < n(j)$.

For each $u \notin V(T^{(2)})$, let u' be the vertex in $T^{(2)}$ of minimum distance from u and let u'' be the vertex in $T^{(2)}$ of minimum distance to u. Note that $n(u') = \frac{d-3}{2}$ or $\frac{d+7}{2}$ and $n(u'') = \frac{d-3}{2}$ or $\frac{d+7}{2}$. Observe also the following facts about H:

(4a) for $u \notin V(T^{(2)})$, $d(u,u') \leq \frac{d-5}{2}$ and $d(u'',u) \leq \frac{d-5}{2}$;

(4b) for $u,v \notin V(T^{(2)})$, $d(u',v'') \leq 5$ (by observations (3c) and (3d)).

Let $u_1,u_2 \notin V(T^{(2)})$ and $v_1,v_2 \in V(T^{(2)})$. By observations (4a) and (4b) above, $d(u_1,u_2) \leq \frac{d-5}{2} + 5 + \frac{d-5}{2} = d$, $d(u_1,v_1) \leq \frac{d-5}{2} + 6 \leq d$ and $d(v_1,u_1) \leq 6 + \frac{d-5}{2} \leq d$. In addition, since $d(F) = 7$, $d(v_1,v_2) \leq 7 \leq d$. Hence $d(H) = d$. Since every vertex in H lies on a cycle of length 4, by Lemma 1, we have the result for $d \equiv 1 (\text{mod } 2)$. □

References

[1] F. Boesch and R. Tindell, Robbin's theorem for mixed multigraphs, *Amer. Math. Monthly* 87 (1980), 716–719.

[2] F.R.K. Chung, M.R. Garey and R.E. Tarjan, Strongly connected orientations of mixed multigraphs, *Networks* 15 (1985), 477–484.

[3] V. Chvátal and C. Thomassen, Distances in orientations of graphs, *J. Combin. Theory Ser. B* 24 (1978), 61–75.

[4] P. Fraigniaud and E. Lazard, Methods and problems of communication in usual networks, *Discrete Applied Math.* 53 (1994), 79–133.

[5] G. Gutin, m-sources in complete multipartite graphs, *Ser. Fiz.-Mat. Navuk.* 5 (1989), 101–106. (In Russian).

[6] G. Gutin, Minimizing and maximizing the diameter in orientations of graphs, *Graphs and Combinatorics* 10 (1994), 225–230.

[7] G. Gutin, Cycles and paths in semicomplete multipartite digraphs, theorems and algorithms: a survey, *J. Graph Theory* 19 (1995), 481–505.

[8] K.M. Koh and B.P. Tan, The diameters of a graph and its orientations, *Res. Rep.* National University of Singapore (1992).

[9] K.M. Koh and B.P. Tan, The diameter of an orientation of a complete multipartite graph, *Discrete Math.* 149 (1996), 131–139.

[10] K.M. Koh and B.P. Tan, The minimum diameter of orientations of complete multipartite graphs, *Graphs and Combinatorics* 12 (1996), 333–339.

[11] K.M. Koh and E.G. Tay, Optimal orientations of products of paths and cycles, *Discrete Applied Math.* 78 (1997), 163–174.

[12] K.M. Koh and E.G. Tay, On optimal orientations of cartesian products of even cycles and paths, *Networks* 30 (1997), 1–7.

[13] K.M. Koh and E.G. Tay, On optimal orientations of cartesian products of graphs (I), *Discrete Math.* 190 (1998), 115–136.

[14] K.M. Koh and E.G. Tay, On optimal orientations of cartesian products of graphs (II): complete graphs and even cycles, *Discrete Math.* 211 (2000), 75–102.

[15] K.M. Koh and E.G. Tay, On optimal orientations of cartesian products of trees, *Graphs and Combinatorics* 17(1) (2001), 79–97.

[16] K.M. Koh and E.G. Tay, On optimal orientations of cartesian products of even cycles, *Networks* 32 (1998), 299–306.

[17] K.M. Koh and E.G. Tay, On optimal orientations of cartesian products with a bipartite graph, *Discrete Applied Math.* 98 (1999), 103–120.

[18] K.M. Koh and E.G. Tay, On optimal orientations of G-vertex multiplications, *Discrete Math.* 219 (2000), 153–171.

[19] J.C. Konig, D.W. Krumme, and E. Lazard, Diameter-preserving orientations of the torus, *Networks* 32 (1998), 1–11.

[20] S.B. Maurer, The king chicken theorems, *Math. Mag.* 53 (1980), 67–80.

[21] J.E. McCanna, Orientations of the n-cube with minimum diameter, *Discrete Math.* 68 (1988), 309–310.

[22] J. Plesník, Diametrically critical tournaments, *Casop. Pest. Matem.* 100 (1975), 361–370.

[23] J. Plesník, Remarks on diameters of orientations of graphs, *Acta Math. Univ. Comenian.* 46/47 (1985), 225–236.

[24] K.B. Reid, Every vertex a king, *Discrete Math.* 38 (1982), 93–98.

[25] H.E. Robbins, A theorem on graphs with an application to a problem of traffic control, *Amer. Math. Monthly* 46 (1939), 281–283.

[26] F.S. Roberts, *Discrete Mathematical Models, with Applications to Social, Biological, and Environmental Problems*, Prentice-Hall, Englewood Cliffs, NJ (1976).

[27] F.S. Roberts, *Graph Theory and its Applications to Problems of Society*, NSF-CBMS Monograph 29, SIAM, Philadelphia, PA (1978).

[28] F.S. Roberts and Y. Xu, On the optimal strongly connected orientations of city street graphs I: Large grids, *SIAM J. Discrete Math.* 1 (1988), 199–222.

[29] F.S. Roberts and Y. Xu, On the optimal strongly connected orientations of city street graphs II: Two east-west avenues or north-south streets, *Networks* 19 (1989), 221–233.

[30] F.S. Roberts and Y. Xu, On the optimal strongly connected orientations of city street graphs III: Three east-west avenues or north-south streets, *Networks* 22 (1992), 109–143.

[31] F.S. Roberts and Y. Xu, On the optimal strongly connected orientations of city street graphs IV: Four east-west avenues or north-south streets, *Discrete Applied Math.* 49 (1994), 331–356.

[32] L. Soltes, Orientations of graphs minimizing the radius or the diameter, *Math Slovaca* 36 (1986), 289–296.

(Received 3 May 2004)