A Conjecture of Goodman and the Multiplicities of Graphs

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Abstract

We prove a conjecture of Goodman [4] about the maximum number of monochromatic triangles in any two-coloring of the edges of K_n with a fixed number of edges in each color.

Moreover, this result is used for finding the multiplicities $M(G;n)$ of some small graphs G, where $M(G;n)$ is defined as the smallest number of monochromatic copies of G in any two-coloring of the edges of K_n.

Together with previous results, this completes the determination of the multiplicities of all graphs with at most three edges.

1. Introduction

In the following, we consider two-colorings of the edges of the complete graph K_n, for short *colorings*, and use red and blue as our colors. For a graph G and a coloring C of K_n, we denote the number of monochromatic copies of G in C by $N_C(G)$ (or just $N(G)$, if it is clear which coloring is referred to).

The *multiplicity* $M(G;n)$ of a graph G and a positive integer n is defined as $\min_C N_C(G)$ over all colorings C of K_n. It includes the *Ramsey number* $r(G)$, which is the smallest n such that $M(G;n)$ is positive, and the *Ramsey multiplicity* $R(G)$, which is $M(G;r(G))$. Those colorings C in which $M(G;n)$ is attained are called *minimizing colorings*.

There are very few exact results about multiplicity: The only graphs G for which $M(G;n)$ was known for all $n \in \mathbb{N}$ were the triangle K_3 (Goodman [3]), the path P_4, and the stars $K_{1,m}$ for all $m \in \mathbb{N}$ (Czerniakiewicz [2], Burr and Rosta [1]). In Section 3, we will determine the multiplicities of $2K_2$, $3K_2$, and $P_3 \cup e$, so that we have the exact values of $M(G;n)$ for all graphs G with at most three edges.

Goodman [4] also determined $\min_C N_C(K_3)$ where the minimum is taken over the colorings C of K_n with some fixed number of red and blue edges. Moreover, he made a conjecture about $\max_C N_C(K_3)$ with the same constraint. (Without this constraint the maximum is trivially attained in a coloring where all edges have the same color.) This conjecture will be proved in Section 2.

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2. The conjecture of Goodman

We will use the following relation between the path P_3 and the triangle K_3: For a given coloring C of K_n, let t be the number of copies of K_3 in C which are not monochromatic. Then we have:

$$\begin{aligned} N_C(K_3) \;+\; t &= \binom{n}{3} \quad \text{and} \\ 3N_C(K_3) \;+\; t &= N_C(P_3), \end{aligned}$$

where the first equation counts all copies of K_3 in C and the second one counts the monochromatic copies of P_3 contained in the copies of K_3 in C. (A generalization of this approach will be discussed later.) This gives:

$$N_C(K_3) = \frac{N_C(P_3) - \binom{n}{3}}{2}, \tag{1}$$

allowing us to rewrite Goodman's conjecture in terms of $N_C(P_3)$ instead of $N_C(K_3)$.

In the following, we denote the number of red edges of a given coloring C of K_n by r. Trivially, $\max_C N_C(P_3)$ over all colorings C with $r = 0$ equals $3\binom{n}{3}$, so we need not consider this case.

For $1 \leq r \leq \binom{n}{2}$, let the integers q, p, ν, and λ be (uniquely) given by the relations

$$r = \binom{q}{2} + q(n-q) + p \quad \text{with} \quad 1 \leq p \leq n-q-1 \quad \text{and} \tag{2}$$

$$r = \binom{\nu}{2} + \lambda \quad \text{with} \quad 0 \leq \lambda \leq \nu - 1. \tag{3}$$

Now two colorings $C_1(n,r)$ and $C_2(n,r)$ of K_n with vertex set $\{v_1, \ldots, v_n\}$ are defined as follows: In $C_1(n,r)$, the vertices $v_1, \ldots, v_q$ form a red K_q and are also joined to each of the remaining vertices $v_{q+1}, \ldots, v_n$ by a red edge. Apart from that, the vertex v_{n-p} is joined to each of the p vertices $v_{n-p+1}, \ldots, v_n$ by a red edge. In $C_2(n,r)$, the vertices $v_{n-\nu}, \ldots, v_n$ form a red $K_{\nu+1}$ where the edges $v_{n-\nu+\lambda}v_i$ have been removed for $i = n-\nu+\lambda+1, \ldots, n$. All other edges in $C_1(n,r)$ and $C_2(n,r)$ are blue.

It follows from (2) and (3) that $C_1(n,r)$ and $C_2(n,r)$ each contain exactly r red edges. Note that $C_1(n,r)$ and $C_2(n, \binom{n}{2} - r)$ are complementary colorings.

With these definitions, what Goodman conjectured can be stated as follows:

Theorem 1. The maximum of $N_C(P_3)$ over all colorings C of K_n with exactly r red edges is attained in $C_1(n,r)$ or $C_2(n,r)$. With q, p, ν, and λ given by (2) and (3), this maximum equals

$$\max \left\{ 3\binom{n}{3} - 2(n-1)r + q(n-1)^2 + (4q+p+1)p + q^2(n-q), \right.$$
$$\left. 3\binom{n}{3} - 2r(n-\nu) + \lambda(\lambda+1) \right\}.$$

Proof. Let C be a coloring of K_n with r red edges. If $r(v_i)$ and $b(v_i)$ denote the red and the blue degrees respectively of a vertex v_i in C, we have:

$$N_C(P_3) = 3\binom{n}{3} - \sum_{i=1}^{n} r(v_i)\, b(v_i) = 3\binom{n}{3} - 2(n-1)r + \sum_{i=1}^{n} r(v_i)^2. \tag{4}$$

Let us call $C_1(n,r)$ and $C_2(n,r)$ the colorings C_1 and C_2 *belonging to* C. We will show that from any coloring C, we can get to one of the colorings C_1 or C_2 belonging to C by edge recolorings in such a way that $N(P_3)$ is not decreased. Therefore $N_{C_1}(P_3)$ or $N_{C_2}(P_3)$ must be the maximum. From the (red) degree sequences of $C_1(n,r)$ and $C_2(n,r)$,

$$(r(v_1),\ldots,r(v_n)) = (\underbrace{n-1,\ldots,n-1}_{q},\underbrace{q,\ldots,q}_{n-q-p-1},q+p,\underbrace{q+1,\ldots,q+1}_{p}) \quad \text{and}$$

$$(r(v_1),\ldots,r(v_n)) = (\underbrace{0,\ldots,0}_{n-\nu-1},\underbrace{\nu,\ldots,\nu}_{\lambda},\lambda,\underbrace{\nu-1,\ldots,\nu-1}_{\nu-\lambda})$$

respectively, it follows that

$$\sum_{i=1}^{n} r(v_i)^2 = q(n-1)^2 + (4q+p+1)p + q^2(n-q) \quad \text{for } C_1(n,r) \text{ and} \tag{5}$$

$$\sum_{i=1}^{n} r(v_i)^2 = 2r(\nu-1) + \lambda(\lambda+1) \quad \text{for } C_2(n,r). \tag{6}$$

(4), (5), and (6) then imply the assertion in the theorem.

In a coloring C of K_n, let $d_{\max} = \max\{r(v_1),\ldots,r(v_n),b(v_1),\ldots,b(v_n)\}$. Let this maximum be attained, for instance, by a red degree, say $d_{\max} = r(v_1)$. Suppose that $d_{\max} < n-1$, i.e. there is a blue edge v_1v for some $v \in V$. Then there must also be a red edge vv' for some $v' \in V$, because otherwise $b(v) = n-1 > d_{\max}$. Now recolor the edge v_1v from blue to red and the edge vv' from red to blue. Then r is unchanged, and $N_C(P_3)$ is increased by

$$r(v_1) + b(v') - (b(v_1)-1) - (r(v')-1).$$

As $r(v_1) = d_{\max} \geq r(v')$ and thus $b(v_1) \leq b(v')$, this expression is positive. Of course, $r(v_1)$ is again the maximum after the recoloring, so this process can be repeated until $r(v_1) = n-1$. Now consider the coloring of K_{n-1} with vertex set $V \setminus \{v_1\}$ and proceed similarly. In this way we obtain a vertex, v_2 say, which has the red or blue degree $n-2$ in this coloring. Repeated application of these arguments eventually yields—if necessary by renumbering the vertices—a coloring C' of K_n still with r red edges and $N_{C'}(P_3) \geq N_C(P_3)$. In C' each vertex v_i is joined to each succeeding vertex v_j $(j > i)$ by edges of the same color.

Let us call a coloring with this property an *order*. Moreover, we will speak of *red* and *blue vertices*, according to the color in which the respective vertex is joined to all succeeding vertices. For an order on n vertices with exactly k red vertices $v_{a_1}, \ldots, v_{a_k}$ $(a_1 < \ldots < a_k)$, write

$$(n \mid a_1, \ldots, a_k).$$

Since v_n can be regarded as a red or a blue vertex, the orders $(n \mid a_1, \ldots, a_{k-1}, n)$ and $(n \mid a_1, \ldots, a_{k-1})$ are identical. Moreover, we have $k \geq 1$, because we only consider colorings with $r \geq 1$ red edges.

Note that colorings of this kind also play a part in a proof of Ramsey's theorem (e.g. see [5], p. 4f.).

With this notation and with q, p, ν, and λ given by (2) and (3), we have:

$$\begin{aligned} C_1(n,r) &= (n \mid 1, \ldots, q, n-p) \quad \text{and} \\ C_2(n,r) &= (n \mid n-\nu, \ldots, n-\nu+\lambda-1, n-\nu+\lambda+1, \ldots, n). \end{aligned}$$

The number of red edges r of an order $(n \mid a_1, \ldots, a_k)$ and of the coloring C_1 belonging to it can be expressed as

$$\sum_{i=1}^{q}(n-i)+p = r = \sum_{i=1}^{k}(n-a_i).$$

Since $a_i \geq i$ for $i = 1, \ldots, k$ and $p \geq 1$, this implies

$$q < k. \tag{7}$$

The most extensive part of the proof is now to show that the value of $N(P_3)$ in an order $(n \mid a_1, \ldots, a_k)$ with r red edges and $k \leq \lfloor n/2 \rfloor$ is not larger than in the coloring C_1 belonging to it. If $k > \lfloor n/2 \rfloor$, we interchange the two colors, so that we have an order with $k \leq \lfloor n/2 \rfloor$ red vertices, and replace r by $\binom{n}{2} - r$. In this case it follows from the complementarity of $C_1(n,r)$ and $C_2(n, \binom{n}{2} - r)$ that the value of $N(P_3)$ in the order is not larger than in the coloring C_2 belonging to it. Together, this implies the assertion.

The following recoloring is essential to all subsequent arguments: In an order $(n \mid a_1, \ldots, a_k)$, choose two red vertices v_{a_i} and v_{a_j} $(1 < a_i < a_j < n)$ such that the vertices v_{a_i-1} and v_{a_j+1} are blue. By definition of an order, the edge $v_{a_i-1}v_{a_i}$ is blue and the edge $v_{a_j}v_{a_j+1}$ is red. Now these two edges are recolored to the other color in each case, so that r remains unchanged. In the new coloring the vertices v_{a_i-1} and v_{a_i} as well as v_{a_j} and v_{a_j+1} have merely interchanged their roles with each other, i.e. we now have the order $(n \mid a_1, \ldots, a_i - 1, \ldots, a_j + 1, \ldots, a_k)$. By this recoloring, $N(P_3)$ is increased by

$$\begin{aligned} & r(v_{a_i-1}) + r(v_{a_i}) + b(v_{a_j}) + b(v_{a_j+1}) \\ & -[b(v_{a_i-1}) - 1] - [b(v_{a_i}) - 1)] - [r(v_{a_j}) - 1] - [r(v_{a_j+1}) - 1] \\ = {} & 2[(i-1) + (i-1+n-a_i) + (a_j - j) + (n-j-1)] - 4(n-1) + 4 \\ = {} & 2[(a_j - a_i - 1) - 2(j-i-1)]. \end{aligned} \tag{8}$$

The expression in brackets in (8) corresponds to the number of blue vertices between a_i and a_j minus the number of red vertices between a_i and a_j.

The recoloring described above can be illustrated in the following way: The order $(n \mid a_1, \ldots, a_k)$ is represented by a sequence of n squares numbered 1 to n, and for each red vertex v_{a_i}, a coin is placed on square # a_i. As the orders $(n \mid a_1, \ldots, a_{k-1}, n)$ and $(n \mid a_1, \ldots, a_{k-1})$ are identical, it makes no difference if there is a coin on square # n or not.

Example.

Figure 1. The order $(10 \mid 2, 4, 5, 8, 10)$.

The recoloring then consists in shifting two coins on squares # a_i and a_j apart by one square each, provided squares # $a_i - 1$ and $a_j + 1$ are empty. This is called a *move* (a_i, a_j). As shown above, a move corresponds to the transition from the order $(n \mid a_1, \ldots, a_i, \ldots, a_j, \ldots, a_k)$ to $(n \mid a_1, \ldots, a_i - 1, \ldots, a_j + 1, \ldots, a_k)$. The number of empty squares between the coins moved minus the number of coins between them is called the *balance* of the move. By (8), a move increases $N(P_3)$ by twice its balance. A sequence of moves that can be carried out one after the other is called a *move sequence*; the *balance of the move sequence* is the sum of the balances of its moves.

Every order can be transformed by a (not necessarily unique) move sequence into the coloring C_1 belonging to it: In each move, for example, shift the leftmost coin whose left neighboring square is empty, and the rightmost coin whose right neighboring square is empty, apart. If, at any time, a coin lies on square # n, then it is removed. As soon as no further move is possible, C_1 is attained. Since each coin only moves in one direction (or not at all) during the move sequence, this is the case after finitely many moves.

The balance of a move sequence that transforms a given order $(n \mid a_1, \ldots, a_k)$ into the coloring C_1 belonging to it, is independent of the move sequence actually chosen, because by (8) it only depends on the change of $N(P_3)$. Therefore we call it the *balance of the order*, written $b(n \mid a_1, \ldots, a_k)$. So we have to prove:

$$b(n \mid a_1, \ldots, a_k) \geq 0 \text{ for } k \leq \left\lfloor \frac{n}{2} \right\rfloor. \tag{9}$$

We first show that (9) for $k = \lfloor n/2 \rfloor$ also implies (9) for $k < \lfloor n/2 \rfloor$. To this end, we need the following lemma:

Lemma 1. For $k \geq 2$,

$$b(n \mid 1, a_1, \ldots, a_{k-1}) = b(n - 1 \mid a_1 - 1, \ldots, a_{k-1} - 1).$$

Example.

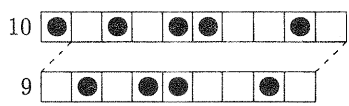


Figure 2.

Proof. The proof is clear if one carries out moves (i,j) and $(i-1,j-1)$ simultaneously in the two orders. $\square$

Now if (9) holds for $k=\lfloor n/2\rfloor$, then for an order $(n\,|\,a_1,\ldots,a_k)$ with $k<\lfloor n/2\rfloor$ by applying Lemma 1 $n-2k-1$ times we have:

$$\begin{aligned} & b(n\,|\,a_1,\ldots,a_k) \\ = \; & b(2n-2k-1\,|\,1,\ldots,n-2k-1,a_1+n-2k-1,\ldots,a_k+n-2k-1) \\ \geq \; & 0, \end{aligned}$$

because the second order contains exactly $n-k-1=\lfloor(2n-2k-1)/2\rfloor$ coins.

For an order $(n\,|\,a_1,\ldots,a_k)$, let $a_i=i$ for $i=1,\ldots,k-t$, $0\leq t\leq k$, and $a_{k-t+1}>k-t+1$. So t is the unique number of coins with the property that there is at least one empty square to the left of them.

Example.

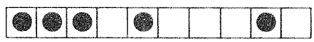
10 

Figure 3. Here, $t=2$.

We now show

$$b(n\,|\,a_1,\ldots,a_k)\geq 0 \text{ for } k=\left\lfloor\frac{n}{2}\right\rfloor \tag{10}$$

by induction on the triple (k,a_1,t), where the linear ordering on this set of triples is given by

$$\begin{aligned} & (k,a_1,t)<(k',a_1',t') \\ \iff \; & (k<k')\vee(k=k'\wedge a_1<a_1')\vee(k=k'\wedge a_1=a_1'\wedge t<t') \end{aligned} \tag{11}$$

(the "lexicographic ordering"). (10) is true for $(k,a_1,t)=(1,1,0)$, because in this case there are only the orders $(2\,|\,1)$ and $(3\,|\,1)$, both of whose balance equals 0.

For the induction step, we need four additional lemmas:

Lemma 2. For $k\leq\lfloor n/2\rfloor$ and $k+1\leq m\leq n$,

$$b(n\,|\,2,\ldots,k,m)\geq 0.$$

Example.

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Figure 4.

Proof. If $n-m\geq k-1$, then the order $(n\,|\,2,\ldots,k,m)$ can be transformed by the move sequence

$$(2,m),(3,m+1),\ldots,(k,m+k-2)$$

into the coloring $C_1 = (n\,|\,1, \ldots, k-1, m+k-1)$ belonging to it. The balance of this move sequence is (use (8))

$$\begin{aligned} b(n\,|\,2, \ldots, k, m) &= \sum_{i=1}^{k-1}[(m-3) - 2(k-1-i)] \\ &= (m-k-1)(k-1) \\ &\geq 0. \end{aligned}$$

If $n - m < k - 1$, let $l = (k-1) - (n-m) \geq 1$, then the order $(n\,|\,2, \ldots, k, m)$ can be transformed by the move sequences

$(2, m), (3, m+1), \ldots, (n-m+1, n-1)$
(now remove the coin on square # n) and, if $l \geq 2$,
$(n-m+2, k), (n-m+3, k+1), \ldots, (k-1, k+l-2)$

into the coloring $C_1 = (n\,|\,1, \ldots, k-2, k+l-1)$ belonging to it. This is possible because $k + l - 1 = 2k - n + m - 2 \leq m - 2 < n$. For the balances of these move sequences, we have:

$$\begin{aligned} b(n\,|\,2, \ldots, k, m) &= \sum_{i=1}^{n-m}[(m-3) - 2(k-1-i)] + \sum_{i=1}^{l-1}[(l-2) - 2(l-1-i)] \\ &= (n-m)(n-2k) \\ &\geq 0. \end{aligned}$$

□

Lemma 3. For $2 \leq k \leq \lfloor n/2 \rfloor$,

$$b(n\,|\,2, a_1, \ldots, a_{k-1}) \geq b(n-2\,|\,a_1 - 2, \ldots, a_{k-1} - 2).$$

Example.

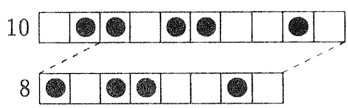


Figure 5.

Proof. Combine Lemmas 1 and 2. □

Lemma 4. For $1 \leq k \leq \lfloor n/2 \rfloor$ and $a_1 \geq 2$,

$$b(n\,|\,a_1, \ldots, a_k) \geq b(n-1\,|\,a_1 - 1, \ldots, a_k - 1).$$

Example.

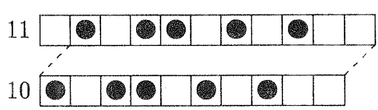


Figure 6.

Proof. Combine Lemmas 1 and 2. □

Lemma 5.

a) For n even, $k = n/2$, and $a_1 \geq 2$,

$$b(n \mid a_1, \ldots, a_k) = b(n+1 \mid a_1 - 1, \ldots, a_k - 1).$$

b) For n odd, $k = (n-1)/2$, and $a_1 \geq 2$,

$$b(n \mid a_1, \ldots, a_k) = b(n+1 \mid a_1 - 1, \ldots, a_k - 1, n).$$

Example.

a)

b)

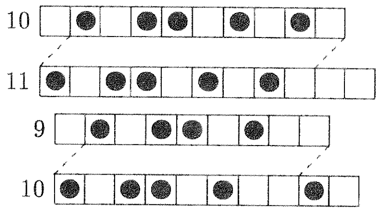


Figure 7.

Proof. Statements a) and b) are proved together by induction on n. In the cases $n = 2$ and $n = 3$, we have:

a) $$b(2 \mid 2) = b(3 \mid 1) = 0 \quad \text{and}$$

b) $$b(3 \mid 2) = b(4 \mid 1, 3) = 0, \quad b(3 \mid 3) = b(4 \mid 2, 3) = 0.$$

Induction step for a): Let n be even, $k = n/2$, and $a_1 \geq 2$. Consider the orders $(n \mid a_1, \ldots, a_k)$ ("order # 1") and $(n+1 \mid a_1 - 1, \ldots, a_k - 1)$ ("order # 2"). We assume that statement b) holds for $n - 1$.

In order # 1, carry out moves of the form (a_1, j_1), $(a_1 - 1, j_2), \ldots$, until the left coin moved lies on square # 2 or no further such move is possible. Carry out the corresponding moves $(a_1 - 1, j_1 - 1)$, $(a_1 - 2, j_2 - 1), \ldots$ in order # 2. Then the difference between the balances of the two orders has not been changed by these moves. Therefore we can assume that the respective move sequence has been applied to orders # 1 and 2.

Case 1. $a_1 = 2$, and there is an i with $2 \leq i \leq k$ and $a_i + 1 \leq n$, such that square # $a_i + 1$ is empty. Then we have:

$$\begin{aligned}
b(n \mid a_1, \ldots, a_k) &= b(n \mid 1, a_2, \ldots, a_i + 1, \ldots, a_k) + (a_i - 2i + 1), \\
&\quad \text{as the move } (2, a_i) \text{ has balance } (a_i - 3) - 2(i-2) \\
&= b(n - 1 \mid a_2 - 1, \ldots, a_i, \ldots, a_k - 1) + (a_i - 2i + 1) \text{ by Lemma 1} \\
&= b(n \mid a_2 - 2, \ldots, a_i - 1, \ldots, a_k - 2, n - 1) + (a_i - 2i + 1) \\
&\quad \text{by induction} \\
&= b(n \mid a_2 - 2, \ldots, a_i - 2, \ldots, a_k - 2, n), \text{ as the move} \\
&\quad (a_i - 1, n - 1) \text{ has balance } (n - a_i - 1) - 2(k - i) \\
&= b(n + 1 \mid 1, a_2 - 1, \ldots, a_k - 1, n + 1) \text{ by Lemma 1} \\
&= b(n + 1 \mid 1, a_2 - 1, \ldots, a_k - 1).
\end{aligned}$$

Case 2. $a_1 > 2$, or for $i = 2, \ldots, k$ either $a_i = n$ or square # $a_i + 1$ is occupied. Because of the initial moves, orders # 1 and 2 then have the form

$$(n \mid m, k+2, \ldots, n) \quad \text{and}$$
$$(n+1 \mid m-1, k+1, \ldots, n-1) \quad \text{respectively with } 2 \le m \le k+1.$$

Here, the assertion is proved directly: Order # 2 is transformed by the move sequences

$$(m-1, n-1), (m-2, n-2), \ldots, (2, n-m+2) \quad \text{and, if } m \le k-1,$$
$$(k+1, n-m+1), (k, n-m), \ldots, (m+2, k+2)$$

first into the order $(n+1 \mid 1, k+1, \ldots, n-m+1, n-m+3, \ldots, n)$ and then into the order $(n+1 \mid 1, m+1, k+3, \ldots, n)$, which (place a coin on square # $n+1$ first) has the same balance as order # 1 by Lemma 1. Moreover, the balances of the two move sequences are

$$\sum_{i=1}^{m-2} [(n-m-1) - 2(k-1-i)] = 0 \quad \text{and} \quad \sum_{i=1}^{k-m} [(k-m-1) - 2(k-m-i)] = 0,$$

which implies the assertion.

Induction step for b): We omit this step because it is essentially the same as for a), only some more cases have to be distinguished. □

Now all lemmas needed for the induction step of the proof of (10) are available. Let an order $(n \mid a_1, \ldots, a_k)$ be given, for which $(k, a_1, t) > (1, 1, 0)$ (in the linear ordering (11)) and $k = \lfloor n/2 \rfloor$ holds. We assume that $b(n' \mid a'_1, \ldots, a'_{k'}) \ge 0$ holds for $k' = \lfloor n'/2 \rfloor$ and $(k', a'_1, t') < (k, a_1, t)$.

Case 1. n even, $a_1 \ge 2$.

$$\begin{aligned} b(n \mid a_1, \ldots, a_k) &= b(n+1 \mid a_1 - 1, \ldots, a_k - 1) \text{ by Lemma 5 a)} \\ &\ge 0 \text{ by induction } (k \text{ equal, } a_1 \text{ smaller}). \end{aligned}$$

Case 2. n even, $a_1 = 1$.

$$\begin{aligned} b(n \mid a_1, \ldots, a_k) &= b(n-1 \mid a_2 - 1, \ldots, a_k - 1) \text{ by Lemma 1} \\ &\ge 0 \text{ by induction } (k \text{ smaller}). \end{aligned}$$

Case 3. n odd, $a_1 \ge 2$.

$$\begin{aligned} b(n \mid a_1, \ldots, a_k) &\ge b(n-1 \mid a_1 - 1, \ldots, a_k - 1) \text{ by Lemma 4} \\ &\ge 0 \text{ by induction } (k \text{ equal, } a_1 \text{ smaller}). \end{aligned}$$

Case 4. n odd, $a_1 = 1$, $a_k \le n-2$.

$$\begin{aligned} b(n \mid a_1, \ldots, a_k) &= b(n-1 \mid 2, a_2 + 1, \ldots, a_k + 1) \text{ by Lemma 5 a)} \\ &\ge b(n-3 \mid a_2 - 1, \ldots, a_k - 1) \text{ by Lemma 3} \\ &\ge 0 \text{ by induction } (k \text{ smaller}). \end{aligned}$$

Case 5. n odd, $a_1 = 1$, $a_k = n-1$.

$$\begin{aligned} b(n \mid a_1, \ldots, a_k) &= b(n+1 \mid 1, 2, a_2+1, \ldots, a_{k-1}+1, n) \text{ by Lemma 1} \\ &= b(n \mid 2, 3, a_2+2, \ldots, a_{k-1}+2) \text{ by Lemma 5 b)} \\ &\geq b(n-2 \mid 1, a_2, \ldots, a_{k-1}) \text{ by Lemma 3} \\ &\geq 0 \text{ by induction } (k \text{ smaller}). \end{aligned}$$

Case 6. n odd, $a_1 = 1$, $a_{k-1} \leq n-2$, $a_k = n$.

$$\begin{aligned} b(n \mid a_1, \ldots, a_k) &= b(n+2 \mid 1, 2, 3, a_2+2, \ldots, a_{k-1}+2, n+2) \\ &\quad \text{by Lemma 1 (applied twice)} \\ &= b(n+2 \mid 1, 2, 3, a_2+2, \ldots, a_{k-1}+2) \\ &= b(n+1 \mid 2, 3, 4, a_2+3, \ldots, a_{k-1}+3) \text{ by Lemma 5 a)} \\ &\geq b(n-1 \mid 1, 2, a_2+1, \ldots, a_{k-1}+1) \text{ by Lemma 3} \\ &= b(n-2 \mid 1, a_2, \ldots, a_{k-1}) \text{ by Lemma 1} \\ &\geq 0 \text{ by induction } (k \text{ smaller}). \end{aligned}$$

Case 7. n odd, $a_1 = 1$, $a_{k-1} = n-1$, $a_k = n$.

$$\begin{aligned} b(n \mid a_1, \ldots, a_k) &= b(n+3 \mid 1, 2, 3, 4, a_2+3, \ldots, a_{k-2}+3, n+2, n+3) \\ &\quad \text{by Lemma 1 (applied three times)} \\ &= b(n+3 \mid 1, 2, 3, 4, a_2+3, \ldots, a_{k-2}+3, n+2) \\ &= b(n+2 \mid 2, 3, 4, 5, a_2+4, \ldots, a_{k-2}+4) \text{ by Lemma 5 b)} \\ &= b(n \mid 1, 2, 3, a_2+2, \ldots, a_{k-2}+2) \text{ by Lemma 3} \\ &\geq 0 \text{ by induction } (k \text{ equal}, a_1 \text{ equal}, t \text{ smaller}). \end{aligned}$$

This implies (10) and therefore, as shown above, also (9), which completes the proof.

□

3. The multiplicities of all graphs with at most three edges

Figure 8 shows all nontrivial graphs with at most three edges and no isolated vertices. Recall that the multiplicities of P_3, K_3, $K_{1,3}$, and P_4 are already known; we will give short new proofs for the last three of them and determine the multiplicities of the remaining three graphs, $2K_2$, $3K_2$, and $P_3 \cup e$.

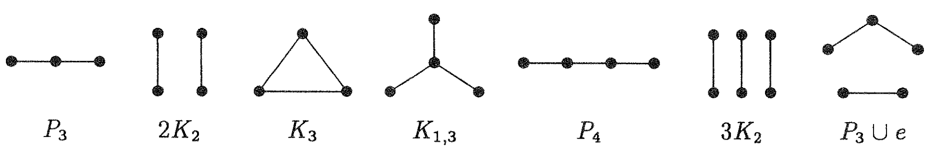


Figure 8.

Theorem 2.

$$M(2K_2;n) = 3\binom{q}{4} + \binom{q}{2}(q-2)(n-q) + 2\binom{q}{2}\binom{n-q}{2} + 3\binom{n-q}{4} \tag{12}$$

$$\text{with } q = \left\lceil n - \frac{5}{2} - \frac{1}{2}\sqrt{2n^2 - 10n + 13} \right\rceil . \tag{13}$$

Proof. With the notation from the proof of Theorem 1, let us call an order of the form $(n\,|\,1,\ldots,q)$ a *regular coloring*. The blue edges in it form a K_{n-q} and the red ones form a $K_q + \overline{K}_{n-q}$. By considering the four possible cases for the monochromatic copies of $2K_2$, we obtain the expression in (12) as the value of $N(2K_2)$ in a regular coloring $(n\,|\,1,\ldots,q)$.

Let r and b again denote the numbers of red and blue edges respectively in a coloring C of K_n. If we consider only the colorings C with some fixed r, then it follows from Theorem 1 and $N_C(P_3) + N_C(2K_2) = \binom{r}{2} + \binom{b}{2}$ that $N(2K_2)$ is the minimum in one of the colorings C_1 or C_2 belonging to them. Therefore (because of the complementarity of $C_1(n,r)$ and $C_2(n,\binom{n}{2}-r)$), we need only consider the orders $(n\,|\,1,\ldots,q,d)$ with $0 \le q \le n-1$ and $q+1 \le d \le n$.

We show that $N(2K_2)$ is not increased during the transition from a coloring of this form to the regular coloring $(n\,|\,1,\ldots,q)$, where q is defined by (13). Hence this regular coloring is a minimizing coloring.

Let the colored K_n have vertex set $V = \{v_1,\ldots,v_n\}$. If the edge $v_d v_{d+1}$ in the order $(n\,|\,1,\ldots,q,d)$ is recolored from red to blue, i.e. during the transition to $(n\,|\,1,\ldots,q,d+1)$, then $N(2K_2)$ is increased by the number of blue edges minus the number of red edges in $(n\,|\,1,\ldots,q,d)$ restricted to $K_n \setminus \{v_d, v_{d+1}\}$, which is a regular coloring $(n-2\,|\,1,\ldots,q)$. This difference is

$$\binom{n-2-q}{2} - \binom{q}{2} - q(n-2-q) = q^2 + (5-2n)q + \frac{(n-2)(n-3)}{2}. \tag{14}$$

Trivially, $M(2K_2;n) = 0$ for $n \le 3$, and $M(2K_2;4) = 0$ is shown by the order $(4\,|\,1)$. Now let $n \ge 5$. Since $q \in \{0,\ldots,n-1\}$, the expression in (14) is greater than or equal to zero for

$$q \le q_1 = \left\lfloor n - \frac{5}{2} - \frac{1}{2}\sqrt{2n^2 - 10n + 13} \right\rfloor$$

and less than or equal to zero for

$$q \ge q_2 = \left\lceil n - \frac{5}{2} - \frac{1}{2}\sqrt{2n^2 - 10n + 13} \right\rceil .$$

If $q \le q_1$ and $q_2 = q_1 + 1$, then one reaches the regular coloring $(n\,|\,1,\ldots,q_2)$ via the orders $(n\,|\,1,\ldots,q',d')$ with $(q',d') = (q,d),(q,d-1),\ldots,(q,q+1),(q+1,n)$, $(q+1,n-1),(q+1,n-2),\ldots,(q+1,q+2),\ldots,(q_1,q_1+1)$, whereby $N(2K_2)$ is not increased. If $2n^2 - 10n + 13$ is a square number, i.e. $q_2 = q_1$, then the desired regular coloring is already reached when $(q',d') = (q_1 - 1, q_1)$. But then the expression in

(14) is equal to zero for $q = q_1$, which means that all the orders $(n \mid 1, \ldots, q_1, d')$ with $d' = q_1 + 1, \ldots, n$ are minimizing colorings.

If $q \geq q_2$, then one reaches the regular coloring $(n \mid 1, \ldots, q_2)$ via the orders $(n \mid 1, \ldots, q', d')$ with $(q', d') = (q, d), (q, d+1), \ldots, (q, n), (q-1, q), (q-1, q+1), \ldots, (q-1, n), \ldots, (q_2 - 1, q_2)$, whereby $N(2K_2)$ is not increased either. □

For determining the multiplicities of further graphs G, we generalize the method that already led to (1):

Consider any coloring C of K_n. Let $T_0, \ldots, T_k$ be the non-isomorphic two-colorings of G. In particular, let T_0 be the coloring where all edges have the same color. Denote the number of copies of G in C colored according to T_i by $t_0, \ldots, t_k$. So we have $t_0 = N_C(G)$.

Moreover, let $H_1, \ldots, H_{k+1}$ be $k+1$ non-isomorphic proper subgraphs of G. $E(H_j)$ denotes the number of "extensions" of H_j to G in K_n, i.e. the number of copies of G that contain a fixed copy of H_j. Finally, let $M_i(H_j)$ be the number of monochromatic copies of H_j contained in a graph G colored according to T_i.

Now if we count, for each monochromatic copy of H_j in C, all of its extensions to G, then each copy of G colored according to T_i is counted exactly $M_i(H_j)$ times, so we have:

$$\sum_{i=0}^{k} M_i(H_j)\, t_i = E(H_j)\, N_C(H_j) \quad \text{for } j = 1, \ldots, k+1. \tag{15}$$

It is convenient to divide the equation for $H_j = K_2$ by the number of edges of G (which is the value of $M_i(K_2)$ for all i) and then interpret it as

$$\sum_{i=0}^{k} t_i = (\text{number of copies of } G \text{ in } K_n).$$

This approach leads to

Theorem 3. If the system of linear equations (15) in $t_0, \ldots, t_k$ has a unique solution, $N_C(G) = t_0$ is obtained as a linear combination of $N_C(H_1), \ldots, N_C(H_{k+1})$.

Our first application of Theorem 3 was (1), where the following non-isomorphic colorings of K_3 were considered (solid and dashed lines represent the two colors):

T_0 T_1

Figure 9.

Taking the minimum on both sides of (1) gives

Theorem 4.

$$M(K_3; n) = \frac{M(P_3; n) - \binom{n}{3}}{2}.$$

Here and in the following theorems, $M(P_3;n)$ can be replaced by (see [1])

$$M(P_3;n) = \begin{cases} \frac{n(n-2)^2}{4} & \text{for } n \equiv 0 \bmod 2 \\ \frac{n(n-1)(n-3)}{4} & \text{for } n \equiv 1 \bmod 4 \\ \frac{n(n-1)(n-3)}{4} + 1 & \text{for } n \equiv 3 \bmod 4. \end{cases}$$

Theorem 5.

$$M(K_{1,3};n) = \frac{(n-3)M(P_3;n) - 4\binom{n}{4}}{2}.$$

Proof. There are two non-isomorphic colorings of $K_{1,3}$:

Figure 10.

Here, the system of equations (15) reads:

$$\begin{array}{rcrcl} t_0 & + & t_1 & = & 4\binom{n}{4} \\ 3t_0 & + & t_1 & = & (n-3)N_C(P_3). \end{array}$$

Again, in in the resulting equation for $N_C(K_{1,3}) = t_0$, the minimum is taken on both sides. □

Theorem 6.

$$M(P_4;n) = \begin{cases} 0 & \text{for } n \leq 4 \\ (n-5)M(P_3;n) + 2\left[\binom{\lfloor\binom{n}{2}/2\rfloor}{2} + \binom{\lceil\binom{n}{2}/2\rceil}{2}\right] - 6\binom{n}{4} & \text{for } n \geq 5. \end{cases}$$

Proof. The order $(4\,|\,1)$ contains no monochromatic P_4, so $M(P_4;n) = 0$ for $n \leq 4$. Now let $n \geq 5$ and consider a coloring C of K_n with r red and b blue edges. There are three non-isomorphic colorings of P_4:

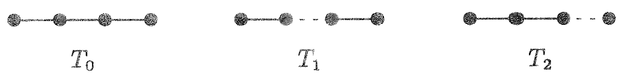


Figure 11.

The system of equations (15) reads:

$$\begin{array}{rcrcrcl} t_0 & + & t_1 & + & t_2 & = & 12\binom{n}{4} \\ t_0 & + & t_1 & & & = & 4\,N_C(2K_2) \\ 2t_0 & & & + & t_2 & = & 2(n-3)\,N_C(P_3). \end{array}$$

This gives

$$N_C(P_4) = t_0 = (n-3)N_C(P_3) + 2N_C(2K_2) - 6\binom{n}{4}$$

and so, as $N_C(P_3) + N_C(2K_2) = \binom{r}{2} + \binom{b}{2}$:

$$N_C(P_4) = (n-5)N_C(P_3) + 2\left[\binom{r}{2} + \binom{b}{2}\right] - 6\binom{n}{4}.$$

Because of $r + b = \binom{n}{2}$ and Jensen's inequality, $\binom{r}{2} + \binom{b}{2}$ is a minimum if r and b both are as close to $\binom{n}{2}/2$ as possible, so the minimum is $\binom{\lfloor \binom{n}{2}/2 \rfloor}{2} + \binom{\lceil \binom{n}{2}/2 \rceil}{2}$. Moreover, there are colorings C of K_n satisfying this condition and minimizing $N_C(P_3)$ at the same time, e.g.: For $n \equiv 0 \pmod 2$, let the red edges form a $K_{n/2,n/2}$ where a matching with $\lfloor n/4 \rfloor$ edges has been removed. For $n \equiv 1, 3 \pmod 4$, let the red edges form a $K_{(n-1)/2,(n-1)/2}$ where a matching with $\lfloor n/4 \rfloor$ edges has been removed and the last vertex has been joined to each vertex of this matching. □

Theorem 7.

$$M(3K_2; n) = \frac{\binom{n-4}{2} M(2K_2; n) - 15\binom{n}{6}}{2}.$$

Proof. There are two non-isomorphic colorings of $3K_2$:

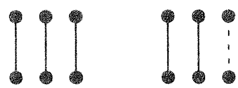


Figure 12.

The proof is similar to that of Theorem 5 with the following system of equations (15):

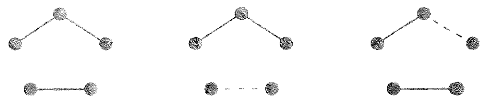
$$\begin{array}{rcl} t_0 + t_1 &=& 15\binom{n}{6} \\ 3t_0 + t_1 &=& \binom{n-4}{2} N_C(2K_2). \end{array}$$

□

Theorem 8.

$$M(P_3 \cup e; n) = \begin{cases} 0 & \text{for } n \le 5 \\ 21, 90, 300, 780, 1683 & \text{for } n = 6,7,8,9,10 \\ \frac{(n-4)(n-11)}{4} M(P_3; n) & \\ \quad +2\left[\binom{\lfloor \binom{n}{2}/2 \rfloor}{2} + \binom{\lceil \binom{n}{2}/2 \rceil}{2}\right] - 15\binom{n}{5} & \text{for } n \ge 11. \end{cases}$$

Proof. There are three non-isomorphic colorings of $P_3 \cup e$:



T_0 T_1 T_2

Figure 13.

The system of equations (15) reads:

$$\begin{array}{rcl} t_0 + t_1 + t_2 &=& 30\binom{n}{5} \\ 2t_0 \qquad\quad + t_2 &=& 4(n-4)N_C(2K_2) \\ t_0 + t_1 \qquad\quad &=& \binom{n-3}{2}N_C(P_3). \end{array}$$

Consider a coloring C of K_n with r red and b blue edges. Then it follows similar to the proof of Theorem 6 that

$$N_C(P_3 \cup e) = \frac{(n-4)(n-11)}{4}N_C(P_3) + 2(n-4)\left[\binom{r}{2} + \binom{b}{2}\right] - 15\binom{n}{5}. \quad (16)$$

Trivially, $M(P_3 \cup e; n) = 0$ for $n \leq 4$. For $n \geq 11$, the coefficient of $N_C(P_3)$ in (16) is non-negative, and the assertion follows similar to the proof of Theorem 6.

And finally, for $n = 5, \ldots, 10$, this coefficient is negative. For the colorings C of K_n with some fixed r and b, the minimum of $N_C(P_3 \cup e)$ is therefore attained in one of the colorings C_1 or C_2 belonging to them by Theorem 1. So for these values of n, we only need to consider the colorings C of the form $(n \mid 1, \ldots, q, d)$. Calculating the respective values of $N_C(P_3 \cup e)$, one finds the orders $(5 \mid 1)$, $(6 \mid 1, 3)$, $(6 \mid 1, 4)$, $(7 \mid 1, 2)$, $(8 \mid 1, 2)$, $(9 \mid 1, 2, 6)$, and $(10 \mid 1, 2, 4)$ as minimizing colorings and the values given in the theorem. □

4. Conclusion

Unfortunately, Theorem 3 fails for graphs G with four edges, e.g. C_4, P_5, or $K_3 + e$: There are either fewer proper subgraphs H_j than non-isomorphic two-colorings T_i of G, or the system of equations (15) does not have a unique solution. Even if the system has a unique solution $N_C(G) = \sum_{j=1}^{k+1} c_j\, N_C(H_j)$, there might not be a coloring in which all terms $c_j\, N_C(H_j)$ are minimized at the same time.

Another open problem is the determination of $M(mK_2; n)$ for $m \geq 4$. It seems that the typical minimizing colorings for these matchings are again certain regular colorings, but it is not clear how to prove this.

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